Designing Profit Shares in Matching and Coalition Formation Games*

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Abstract. Matching and coalition formation are fundamental problems in a variety of scenarios where agents join efforts to perform tasks, such as, e.g., in scientific publishing. To allocate credit or profit stemming from a joint project, different communities use different crediting schemes in practice. A natural and widely used approach to profit distribution is equal sharing, where every member receives the same credit for a joint work. This scheme captures a natural egalitarian fairness condition when each member of a coalition is critical for success. Unfortunately, when coalitions are formed by rational agents, equal sharing can lead to high inefficiency of the resulting stable states.

In this paper, we study the impact of changing profit sharing schemes in order to obtain good stable states in matching and coalition formation games. We generalize equal sharing to sharing schemes where for each coalition each player is guaranteed to receive at least an α-share. This way the coalition formation can stabilize on more efficient outcomes. In particular, we show a direct trade-off between efficiency and equal treatment. If k denotes the size of the largest possible coalition, we prove an asymptotically tight bound of $k^2\alpha$ on prices of anarchy and stability. This result extends to polynomial-time algorithms to compute good sharing schemes. Further, we show improved results for a novel class of matching problems that covers the well-studied case of two-sided matching.

1 Introduction

Matching problems are central to a variety of research at the intersection of computer science and economics. The standard model of matching with preferences is stable matching, in which a set of agents strives to group into pairs, and each agent has an ordinal preference list over all possible partners. In this case, a matching is stable if it has no blocking pair, i.e., no pair of players could both improve by pairing up and dropping their current partners. Applications of this model include, e.g., matching in job markets, hospitals, colleges, social networks, or distributed systems [2, 8, 15, 17, 18]. Numerous extensions of this standard model have been treated in the past [15, 22].

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While the basic stable matching model uses ordinal preferences, many applications allow cardinal preferences to express incentives in terms of profit or reward. Perhaps the most prominent case of cardinal preferences studied in the literature are correlated preferences, in which each matched pair generates a profit that is shared equally among the involved agents. This model has favorable properties, e.g., existence of a stable matching is guaranteed by a potential function argument, and convergence time of improvement dynamics is polynomial [1]. These conditions extend even to hedonic coalition formation games, when instead of matching pairs the agents construct a partition into coalitions of $k > 2$ players. However, these properties come at a cost – the total reward of every stable coalition structure can be up to k times smaller than in optimum.

For projects which need the joint effort of multiple agents to be realized it is difficult to decide how to share the produced value among the participating agents. Oftentimes there is no direct correlation between some player's effort or invested time and the value of the final result. Thus mainly two concepts are used in literature to determine how much is handed to which player. Either all players receive an equal share of the gained value taking into account that all agents were needed to fulfill the project and had to put effort into it or players can negotiate using outside options (e.g. other projects they could take part in instead) to justify their (desired) share. As stated above the main drawback of equal sharing is that it might result in inefficient stable states. On the other hand bargaining might not even reach a stable state and further can result in rather unfair distributions. Our hope is that if some kind of community or ruler can dictate the value-distribution of each project, it is possible to provide more efficient stable outcomes while also having close-to-equal shares. To better understand the conflict between efficient stable states and equal shares we consider the following small example:

Here the agents are given by the vertices while the edges symbolize projects. The edge weight gives the value of the project. Now obviously realizing project $\{1,2\}$ and $\{3,4\}$ would be optimal in this case, but if we choose the equal-sharing rule 2 and 3 will prefer their joint project which makes $\{\{2,3\}\}$ the only stable state. As 1 would gain 0 in such a state in a bargaining solution he might be willing to offer 2 450 of the total 500 their project would be worth to outbid the 400 2 receives from $\{2,3\}$. This way 1 would still get $50 > 0$. The same considerations might motivate 4 to give a higher share to 3. Then the optimal solution $\{\{1,2\},\{3,4\}\}$ is stable, but 1 would only receive 10% of the value his project generates. Even if 1 will only slightly outbid the share of project $\{2,3\}$ he cannot keep more the 20% of the project value for himself. Now if we can dictate the shares we could divide project $\{1,2\}$ such that player 2 gets 308 while 192 remain for 1 and project $\{2,3\}$ such that 307 would go to 2 and 493 to 3. The 2 would want to form $\{1,2\}$ which only leaves $\{3,4\}$ for 3. At the same time both projects are now divided roughly $\frac{5}{13} \approx 38,5\% \setminus \frac{8}{13} \approx 61,5$ and project $\{3, 4\}$ can even be shared equally without jeopardizing the stability of the optimal solution.

In practice we can find equal sharing schemes as well as those which factor in power or effort of the different agents. For agents working on a joint project, equal sharing implements a natural egalitarian fairness condition. For example, in mathematics and theoretical computer science it is common practice to list authors in alphabetical order, which gives equal credit to every author involved in a paper. This is justified by the argument that a ranking of ideas that led to the results in a paper is often impossible. On the other hand, in many other sciences the author sequence gives different credit to the different authors involved in the project. In some cases, these approaches are overruled by the community which gives most credit for a paper to its most prominent author (or to authors that are PhD students). Naturally, such different profit sharing schemes generate different incentives for the agents to form coalitions. In this paper, we study the impact of profit distribution on fairness and efficiency in the resulting coalition formation games. It is known that completely arbitrary sharing can lead to non-existence of stable states or arbitrarily high price of anarchy [3]. Similar to recent work [20], our focus is to design good profit or credit distribution schemes such that the stable states implement good outcomes. In this direction, it is not difficult to observe that using arbitrarily low profit shares we can stabilize every optimal partition. However, such sharing schemes are clearly undesirable when we want to maintain egalitarian fairness conditions. In our analysis, we provide asymptotically tight bounds on the inherent tension between efficiency and equal treatment. Further, we give efficient algorithms to compute good sharing schemes and show complementing hardness results. Before we state our results, we start with a formal description of the model.

1.1 Stable Matchings and Coalition Structures

We assume that there is a simple, undirected graph $G = (V, E)$, where V is the set of *agents* and E the set of possible *projects* or *edges*. In the matching case, we assume each edge is a pair $e = \{u, v\} \in E$ and yields some *profit* $w(e) > 0$ that is to be shared among u and v. Our goal is to design a profit distribution scheme d with $d_u(e), d_v(e) \in [0, 1]$ and $d_u(e) + d_v(e) = 1$ for all $e \in E$. This implies that u gets individual profit $d_u(e)w(e)$ when being matched in e. The profits yield an instance of stable matching with cardinal preferences. The *stable matchings* $M \subset E$ are matchings that allow no *blocking pair* – no pair $\{u, v\} \in E \setminus M$ of agents that can both strictly increase their individual profit by destroying their incident edge in M (if any) and creating $\{u, v\}$. The social welfare of a matching M is $w(M) = \sum_{e \in M} w(e)$. We denote by M^* a (possibly non-stable) optimum matching that maximizes social welfare. The *price of anarchy/stability* (denoted PoA/PoS) is the ratio of $w(M^*)/w(M)$, where M is the worst/best stable matching, respectively. While the set of stable matchings depends on d, the social welfare of a particular matching is independent of d, and so is the set of optimum matchings.

We generalize this scenario to hedonic coalition formation with arbitrary coalition size in a straightforward way. Instead of edges e we are given a set of possible *hyperedges* or *coalitions* $\mathcal{C} \subseteq 2^V$. Each $S \in \mathcal{C}$ fulfills $|S| \geq 2$ and yields profit $w(S) > 0$. The distribution scheme has $d_u(S) \in [0,1]$ and $\sum_{u \in S} d_u(S) = 1$. Then d again specifies the fraction of $w(S)$ allocated to $u \in S$ when coalition S forms. A *coalition structure* $\mathcal{S} \subseteq \mathcal{C}$ is a collection of sets from $\mathcal{C}$ that is mutually disjoint. $\mathcal{S}$ is *core-stable* if there is no *blocking coalition* – no $S \in \mathcal{C} \setminus \mathcal{S}$ of agents that each and all strictly improve their individual profit by destroying their incident coalition in $\mathcal{S}$ (if any) and creating S. Observe that the usual definition of core-stability involves all possible coalitions $S \in 2^V$. We can easily allow this by assuming that $w(\{v\}) = 0$ for all $v \in V$ and $w(S) = -1$ for all $S \in 2^V \setminus \mathcal{C}$ with $|S| > 1$. Definitions of social welfare and prices of anarchy and stability extend in the obvious way. An instance is called *inclusion monotone* if for $S, S' \in \mathcal{C}$ and $S' \subset S$, we have $w(S)/|S| > w(S')/|S'|$. We denote by $k = \max_{S \in \mathcal{C}} |S|$. Stable matchings are exactly core-stable coalition structures when $|S| = 2$ for all $S \in \mathcal{C}$. By definition, every such instance is inclusion monotone.

Our aim is to design d in order to obtain good core-stable coalition structures. To characterize the tension between stability and equal treatment, a distribution scheme is termed *α-bounded* if $d_u(S) \geq \alpha$ for all $S \in \mathcal{C}$ and all $u \in S$. If d is α-bounded, the resulting instance of hedonic coalition formation is termed *α-egalitarian*.

Throughout the paper, we assume a profit of 0 for singleton coalitions, which is in some sense without loss of generality. Suppose we have $\{v\} \in \mathcal{C}$ with $w(\{v\}) > 0$ for a node $v \in V$. Such a player will participate in a coalition only if he receives profit at least $w(\{v\})$. Thus, we can reduce the profit of every coalition $S \in \mathcal{C}$ with $v \in S$ by this amount. By executing this step for every player and every coalition, we obtain an instance with the desired properties. Coalitions that arrive at zero profit in this way can be disregarded, as they can be assumed to be neither part of any equilibrium nor in the optimum solution. Applying our algorithms to the remaining instance, we strive to equally distribute the surplus that the coalition generates over individually required profits. This objective is closely related to Nash bargaining solutions [21]. Note that in the remaining instance with all $w(\{v\}) = 0$, we get larger prices of anarchy and stability. As our bounds apply to all instances of this sort, they continue to hold accordingly for all instances with arbitrary positive $w(\{v\})$.

1.2 Results and Overview

In Section 2 we characterize the effect of α-boundedness on the resulting prices of anarchy and stability. We provide asymptotically tight bounds on prices of anarchy and stability depending on α. Given an optimum coalition structure $\mathcal{S}^*$, we show how to design a distribution scheme d that guarantees (1) existence of a core-stable coalition structure and (2) a price of stability of $\max\{1, k^2\alpha\}$, for any $\alpha \in [0, 1/k]$. This result shows, in particular, that for every $\alpha \leq 1/k^2$, we can construct α-bounded schemes with an optimal core-stable coalition structure. This is asymptotically tight – using α-bounded schemes we cannot achieve a price of stability of less than $(k^2-k)\alpha$, i.e., the price of stability for α-bounded schemes is in $\Theta(k^2)\alpha$. Conversely, this bound translates into a bound on $\alpha \leq \delta/(k^2-k)$ to guarantee price of stability at most δ. For inclusion monotone instances, we can also provide the same upper bound of $\max\{1, k^2\alpha\}$ on the price of anarchy, i.e., in such instances and $\alpha \leq 1/k^2$, every core-stable structure is optimal. In contrast, there exist instances that are not inclusion monotone, in which α-bounded schemes cannot guarantee a price of anarchy of 1, even for arbitrarily small $\alpha > 0$.

While computing $\mathcal{S}^*$ is NP-hard, we can also combine our algorithms with efficient approximation algorithms for the set packing problem of optimizing social welfare. If $\mathcal{S}^*$ is an arbitrary ρ-approximation to the optimum solution, our algorithms can be used to construct in polynomial time an α-bounded distribution scheme that guarantees price of stability of $\rho \cdot \max\{1, k^2\alpha\}$. The same result can be achieved for the price of anarchy in inclusion monotone instances.

In addition, we study a problem inspired by computing core imputations in coalitional games. For a given coalition structure with profits we aim to determine a distribution scheme with largest α that stabilizes a given optimum solution $\mathcal{S}^*$. This problem is shown NP-hard whenever we have coalitions of size $k \geq 3$. The problem remains hard for $k = 2$ if instead of a solution, we have a given bound W, and the goal is to maximize α such that at least one solution of social welfare at least W is stable.

In Section 3 we study stable matching games. As the general results from the previous section carry over, we concentrate on a subclass of instances that we term acyclic alternating. This includes the standard case of bipartite matching. In this case, we can show that even 1/3-bounded distribution schemes yield a PoS of 1. In addition, given an instance and any solution M, an α-bounded distribution scheme stabilizing M with maximal α can be found efficiently.

1.3 Related Work

We study profit distribution in cardinal stable matching and more general games. Stable matching has been extensively studied [22] and the literature on the problem is too vast to survey here. Directly related to our work are [4, 5] which address the price of anarchy in stable matching and related models. Very recently, we have studied the price of anarchy under different edge-based profit sharing schemes [3]. In contrast, this paper concentrates on designing profit shares to guarantee good stable matchings.

Profit sharing in more general coalition formation games has been studied recently [6] in a related model, where coalitions are represented by resources. Agents can join and leave a resource/coalition unilaterally. The authors focus on submodular profit functions and three particular sharing schemes. For the resulting games, they derive results on existence of pure Nash equilibrium, price of anarchy, and convergence of improvement dynamics. In contrast to this model, we do not restrict the number of coalitions that can be formed simultaneously and assume coalitional deviations and core-stability.

In cooperative game theory, profit sharing has been a major focus over the last decades. For example, core stability in the classic transferable-utility cooperative matching game assumes that the total profit of a global maximum matching is distributed to all agents such that every subset S of agents receives in sum at least the value of a maximum weight matching for S. Computing such imputations is closely related to LP duality [13]. Computing different solution concepts in this game has also been of interest [7, 12, 19]. In contrast, we assume utility transfer only within coalitions and evaluate the quality of a scheme based on the price of anarchy for coalitional stability concepts in the resulting coalition formation game. Additionally, we focus on trade-offs between efficiency and equality.

Computing stability concepts in hedonic coalition formation games is a recent line of research in computational social choice [10, 16]. Many stability concepts are NP- or PLS-hard to compute. This holds even in the case of additive-separable coalition profits, which can be interpreted by an underlying graph structure with weighted edges, and the profit of a coalition is measured by the total edge weights covered by the coalition [14, 23]. In addition, some price of anarchy results recently appeared in [9]. While our main focus are structures inspired by matching problems, designing profit shares in the additive-separable case can be formulated in our model, and it represents an interesting avenue for future work.

Designing good cost sharing schemes to minimize prices of anarchy and stability [11, 24] in resulting strategic games is a topic of recent interest in algorithmic game theory.

2 Coalition Formation

We start by analyzing the relation between α-boundedness and the PoS/PoA. At first we will see that we can give non-trivial upper bounds on the PoS and PoA subject to α. In addition, given an optimum solution we can compute a distribution scheme that obtains these bounds.

Theorem 1. *For any $\alpha \in \left[0, \frac{1}{k}\right]$, there is a distribution scheme $d(\alpha)$ that is α-bounded and results in a PoS of at most $\max\{1, k^2\alpha\}$. If further the instance is inclusion monotone, the distribution scheme ensures a PoA of at most $\max\{1, k^2\alpha\}$. Given any social optimum $\mathcal{S}^*$, we can compute the distribution scheme in polynomial time.*

Proof. We provide algorithms that compute the suitable distribution schemes. For the PoS see Algorithm 1 and for the PoA see Algorithm 2. The idea of both algorithms to always consider the worthiest remaining coalition S and use it to decide which coalition S_i^* to stabilize next. If S is part of the optimal coalition structure $\mathcal{S}^*$ we make it S_i^*. Otherwise, if S is overlapping with some worthy enough coalition S' of $\mathcal{S}^*$, we pick S' as S_i^*. Thus in both cases we stabilize an edge of the optimal coalition structure. If the coalition S is not in $\mathcal{S}^*$ but too worthy to be outbid by a $\frac{1}{k}$-share of some overlapping coalition of $\mathcal{S}^*$, we set $S = S_i^*$ instead. As that only happens when the value difference is quite big and

Algorithm 1: Ensuring PoS

Data: Instance $(N, \mathcal{C}, w)$, social optimum $\mathcal{S}^*$, bound α

1 set $i = 0$, $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{S} = \emptyset$;
2 **while** $\mathcal{C}_i \neq \emptyset$ **do**
3   choose S with $w(S) = max\{w(S) \mid S \in \mathcal{C}_i\}$;
4   set $i = i + 1$;
5   **if** $S \in \mathcal{S}^*$ **then**
6   set $S_i^* = S$;
7   **else if** $S \notin \mathcal{S}^*$ *and* $\alpha w(S) < \frac{1}{k} w(S')$ *for some* $S' \in \mathcal{S}^*$ **then**
8   choose $S' \in \mathcal{S}^*$ with $\alpha w(S) < \frac{1}{k} w(S')$;
9   set $S_i^* = S'$;
10   **else**
11   set $S_i^* = S$;
12   set $\mathcal{C}_i = \mathcal{C}_{i-1} \setminus \{S_i^*\}$ and $\mathcal{S} = \mathcal{S} \cup \{S_i^*\}$;
13   **foreach** $u \in S_i^*$ **do**
14   set $d_u(S_i^*) = \frac{1}{|S_i^*|}$;
15   **foreach** $S' \in \mathcal{C}_i$ *with* $S' \cap S_i^* \neq \emptyset$ **do**
16   choose $u \in S' \cap S_i^*$ and set $d_u(S') = \alpha$;
17   **foreach** $u' \in S' \setminus \{u\}$ **do**
18   set $d_{u'}(S') = \frac{1-\alpha}{|S'|}$;
19   set $\mathcal{C}_i = \mathcal{C}_i \setminus \{S'\}$;

the number of affected optimal coalitions per occurrence is limited, this way we get a good bound on how much we loose against the optimum. In $\mathcal{S}$ we keep track of the stable solution. To ensure that S_i^* is stable, $w(S_i^*)$ is shared equally and all overlapping coalitions such that the players joint with S_i^* only receive an α-share.

We start with proving that in both cases $\mathcal{S}$ is core-stable. Obviously $\mathcal{S}$ is a coalition structure. The crucial point for the algorithms to work is as follows. All coalitions S' distributed in round i are of value at most $w(S)$ for the initially chosen S of round i. Hence, for every S' of round i at least one of the players they share with S_i^* wants to stay at S_i^* as by choice of S_i^* $\frac{1}{|S_i^*|} w(S_i^*) \geq \frac{1}{k} w(S_i^*) > \alpha w(S')$. Furthermore, in Algorithm 2 for all coalitions S' of round i the players they share with S_i^* actually prefer S_i^* for the same reason. Then obviously $\mathcal{S}$ is stable as for every $S^+ \in \mathcal{C} \setminus \mathcal{S}$ we have $S^+ \in \mathcal{C}_{i-1} \setminus \mathcal{C}_i$ for some i. Hence there is some agent in $S^+ \cap S_i^*$ (namely u of Line 16 in Algorithm 1 respectively Line 17/Line 21 in Algorithm 2) which refuses to deviate from S_i^* to S^+.

For Algorithm 2 we show that further there is no other core-stable state under $d(\alpha)$. To see this assume some other coalition structure $\mathcal{S}'$ and consider some coalition S^+ of $\mathcal{S} \setminus \mathcal{S}'$ with i minimal such that $S^+ \in \mathcal{C}_{i-1} \setminus \mathcal{C}_i$. Now $S^+ = S_i^*$ and all coalitions of $\mathcal{S}'$ which intersect with S^+ where distributed in the same round i – as otherwise there would have been a coalition of an earlier round in $\mathcal{S} \setminus \mathcal{S}'$. Thus all involved players want to deviate to S_i^* because they are either unmatched or get worse profit from their coalition in $\mathcal{S}'$ than from S_i^*. Here it is

Algorithm 2: Ensuring PoA

Data: Instance $(N, \mathcal{C}, w)$, $\mathcal{C}$ inclusion monotone, social optimum S^*, bound α

1 set $i = 0$, $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{S} = \emptyset$;
2 **while** $\mathcal{C}_i \neq \emptyset$ **do**
3   choose S with $w(S) = max\{w(S) \mid S \in \mathcal{C}_i\}$;
4   set $i = i + 1$;
5   **if** $S \in \mathcal{S}^*$ **then**
6   set $S_i^* = S$;
7   **else if** $S \notin \mathcal{S}^*$ *and* $\alpha w(S) < \frac{1}{k} w(S')$ *for some* $S' \in \mathcal{S}^*$ **then**
8   choose $S' \in \mathcal{S}^*$ with $\alpha w(S) < \frac{1}{k} w(S')$;
9   set $S_i^* = S'$;
10   **else**
11   set $S_i^* = S$;
12   set $i = i + 1$, $\mathcal{C}_i = \mathcal{C}_{i-1} \setminus \{S_i^*\}$ and $\mathcal{S} = \mathcal{S} \cup \{S_i^*\}$;
13   **foreach** $u \in S_i^*$ **do**
14   set $d_u(S_i^*) = \frac{1}{|S_i^*|}$;
15   **foreach** $S' \in \mathcal{C}_i$ *with* $S' \cap S_i^* \neq \emptyset$ **do**
16   **if** $S' \subset S^*$ **then**
17   **foreach** $u \in S'$ **do**
18   set $d_u(S') = \frac{1}{|S'|}$;
19   **else**
20   **foreach** $u \in S'$ **do**
21   **if** $u \in S_i^*$ **then**
22   set $d_u(S') = \alpha$;
23   **else**
24   set $d_u(S') = \frac{1 - \alpha |S_i^* \cap S'|}{|S' \setminus S_i^*|}$;
25   set $\mathcal{C}_i = \mathcal{C}_i \setminus \{S'\}$;

important that the profits are inclusion monotone to ensure that all the players of subcoalitions of S_i^* and not just some want to switch due to higher profits from S_i^*(Line 17).

Hence, we can use $\mathcal{S}$ to give an upper bound on the PoS respectively the PoA. We compare $\mathcal{S}$ to the optimal outcome $\mathcal{S}^*$ we used for the algorithm. For each coalition in $\mathcal{S} \cap \mathcal{S}^*$ both structures give the same value. Next we assign each coalition $S \in \mathcal{S}^* \setminus \mathcal{S}$ to the coalition S_i^* for i such that $S \in \mathcal{C}_{i-1} \setminus \mathcal{C}_i$. Now each S_i^* has at most k coalitions S assigned to it as the size of S_i^* limits the number of mutually disjoint coalitions intersecting with S_i^*. Further, by the choice of S_i^* each of the Ss fulfills $\alpha w(S) < \frac{1}{k} w(S_i^*)$. That is, $\mathcal{S}$ looses at most $k^2 \alpha w(S_i^*)$ compared to $\mathcal{S}^*$ for every coalition S_i^* in $\mathcal{S} \setminus \mathcal{S}^*$. This gives us a PoS and a PoA at most $k^2\alpha$. □

The previous proof can be applied directly even if $\mathcal{S}^*$ is not an optimum solution. Optimality of $\mathcal{S}^*$ only served to establish a relation to the optimum value for social welfare. Hence, if we run Algorithms 1 and 2 on a ρ-approximate solution $\mathcal{S}$, we obtain core-stable states for which social welfare is at most $k^2\alpha$ worse than

$w(\mathcal{S})$. This allows to obtain α-bounded distribution schemes with bounded PoS and PoA in polynomial time.

Corollary 1. *Given any coalition structure $\mathcal{S}$ that is a ρ-approximation to the optimum, Algorithm 1 computes an α-bounded distribution scheme such that the PoS is at most $\rho \cdot \max\{1, k^2\alpha\}$. The same result holds for Algorithm 2 and PoA in inclusion monotone instances.*

Now, while we had to make no assumptions about the profits to ensure the bounds on the PoS for the PoA we asked for w to be inclusion monotone. The question arises whether this limitation was solemnly due to the structure of our algorithm or if non-inclusion monotone instances do not allow the same bounds on the PoA. It turns that while we can get arbitrary close to and even reach 1 for the PoS and the PoA of inclusion monotone instances if we just make α small enough, we can find non-inclusion monotone instances where the PoA is at least $2 - \epsilon$ independent of the choice of α. That is, even if we allow α to drop to zero basically allowing to give nothing to some (participating) players there can still be suboptimal stable states.

Proposition 1. *There exist instances with n agents and w not inclusion monotone, in which the PoA is at least $2 - \frac{4}{n+2}$ for every distribution scheme.*

Proof. Consider the instance $N = \{1, \dots, n\}$ with profits $w(N) = n$, $w(S) = \lceil \frac{n+1}{2} \rceil$ for every $S \subset N$ with $|S| = \lceil \frac{n+1}{2} \rceil$ and $w(S) = 0$ for every other $S \subset N$. The unique social optimum is the grand coalition $S = N$ as each other coalition structure can hold at most 1 set of size $\lceil \frac{n+1}{2} \rceil$. Assume that there is some distribution scheme d which ensures that N is stable. Consider the set S of the $\lceil \frac{n+1}{2} \rceil$ players which get the lowest shares. We have to destabilize all coalitions of size $\lceil \frac{n+1}{2} \rceil$. Hence, the grand coalition has to offer in total at least $\lceil \frac{n+1}{2} \rceil + \epsilon$ with arbitrarily small $\epsilon > 0$ to each group of $\lceil \frac{n+1}{2} \rceil$ players. Then at least one of the players in S gets at least

$$\frac{\lceil \frac{n+1}{2} \rceil + \epsilon}{\lceil \frac{n+1}{2} \rceil}$$

and, by the choice of S, every player in $N \setminus S$ as well. Thus, N distributes total profits of at least

$$(\left\lceil \frac{n+1}{2} \right\rceil + \epsilon) + (n - \left\lceil \frac{n+1}{2} \right\rceil) \cdot \left(\frac{\lceil \frac{n+1}{2} \rceil + \epsilon}{\lceil \frac{n+1}{2} \rceil} \right) > n$$

a contradiction. □

Next we want to analyze how close the values provided by our algorithms are to the actual bounds. Note that for the extreme of equal sharing of the biggest coalitions $\alpha = \frac{1}{k}$, we get an upper bound of k for the PoS and the PoA while for $\alpha = \frac{1}{k^2}$ we reach PoS = PoA = 1. In particular for every $\alpha \leq \frac{1}{k^2}$ we can always assure optimality of core-stable coalition structures.

Conversely, for all k and $\alpha = \frac{1}{k}$ we can show tightness through the example $N = \{1, \ldots, k^2\}$ with $w(\{1, \ldots, k\}) = 1 + \epsilon$, $w(\{i + jk \mid j = 0 \ldots k-1\}) = 1$, $i = 1 \ldots k$ and $w(S) = 0$ for every other coalition S. For $\epsilon \to 0$ this leads to a PoS of k.

Furthermore our algorithms provide asymptotically tight bound for all choices of α as there are instances of α-egalitarian games where PoS $\in \Theta(k^2)\alpha$.

Proposition 2. *For every $k > 2$, there is an instance in which every α-bounded distribution scheme yields a PoS of at least* $\max\{1, (k^2 - k)\alpha\}$.

Proof. Consider the following instance with $N = \{v_{i,j} \mid i = 1 \ldots k, j = 1 \ldots k-1\} \cup \{v_0\}$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}_1 = \{S_j = \{v_{i,j} \mid i = 1 \ldots k\} \mid j = 1 \ldots k-1\}$ and $\mathcal{C}_2 = \{\{v_0\} \cup \{v_{i_j,j} \mid j = 1 \ldots k-1\} \mid j = 1 \ldots k-1, j_i = 1 \ldots k\}$. For $S \in \mathcal{C}_1$ $w(S) = 1$ and for $S \in \mathcal{C}_2$ we have $w(S) = \frac{1+\epsilon}{k\alpha}$ for some small enough ϵ. Note that each single coalition of $\mathcal{C}_2$ covers every other coalition of $\mathcal{C}_2$ via v_0 and every coalition S_j of $\mathcal{C}_1$ via some $v_{\cdot,j}$. Thus for $\alpha < \frac{1}{k^2-k}$ $\mathcal{C}_1$ with a total value of $k-1$ is the unique social optimum. Regardless of the distribution scheme on $\mathcal{C}_1$ there is always coalition C in $\mathcal{C}_2$ which meets an offer of at most $\frac{1}{k}$ from every coalition in $\mathcal{C}_1$. Thus, even if we share C such that $d_{v_0}(C) = 1 - (k-1)\alpha$ and $d_v = \alpha$ for every other $v \in C$, we still have an offer of $\le \frac{1}{k}$ versus $\alpha(\frac{1+\epsilon}{k\alpha}) = \frac{1+\epsilon}{k}$ at every vertex C shares with some coalition of $\mathcal{C}_1$, that is, C destabilizes $\mathcal{C}_1$. Hence the PoS is

$$\frac{k-1}{\frac{1+\epsilon}{k\alpha}} = \frac{k-1}{\frac{(1+\epsilon)(k-1)}{k(k-1)\alpha}} \xrightarrow{\epsilon \to 0} (k^2 - k)\alpha. \quad \square$$

Remark. *In reverse, there are instances where for PoS at most δ the required α is at least $\frac{\delta}{k^2-k}$. Our algorithms compute a distribution scheme for $\alpha = \frac{\delta}{k^2}$.*

We next consider deciding if a given α is small enough to allow for a distribution scheme with a core-stable coalition structure that obtains guaranteed total profit. Equivalently, we consider finding the smallest α such that an α-bounded distribution scheme yields a stable structure with a certain social welfare.

Theorem 2. *It is* NP*-hard to decide whether for a given $\alpha > 0$ and a given value $W > 0$ there is an α-bounded distribution scheme that admits a core-stable coalition structure $\mathcal{S}$ such that $\sum_{S \in \mathcal{S}} w(S) \ge W$. This holds even for instances with $k = 2$.*

Proof.

For the proof we will show a reduction of 3Sat and use only coalitions of size 2. Given a 3Sat formula with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$, where clause C_j contains the literals $l1_j, l2_j$ and $l3_j$, we have $N = \{x_i, \overline{x}_i \mid i = 1 \ldots n\} \cup \{C_{j,1,1}, C_{j,1,2}, C_{j,2,1}, C_{j,2,2}, C_{j,3,1}, C_{j,3,2} \mid j = 1 \ldots m\}$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ where $\mathcal{C}_1 = \{\{x_i, \overline{x}_i\} \mid i = 1 \ldots n\}$, $\mathcal{C}_2 = \{\{C_{j,1,1}, C_{j,1,2}\}, \{C_{j,2,1}, C_{j,2,2}\}, \{C_{j,3,1}, C_{j,3,2}\} \mid j = 1 \ldots m\}$, $\mathcal{C}_3 = \{\{C_{j,1,1}, C_{j,2,1}\}, \{C_{j,2,1}, C_{j,3,1}\}, \{C_{j,3,1}, C_{j,1,1}\} \mid j = 1 \ldots m\}$ and $\mathcal{C}_4 = \{\{C_{j,1,1}, l1_j\}, \{C_{j,2,1}, l2_j\}, \{C_{j,3,1}, l3_j\} \mid j = 1 \ldots m\}$. For the values we have $w(S) = 10$ for $S \in \mathcal{C}_1$, $w(S) = 9$ for $S \in \mathcal{C}_2$, $w(S) = 100$ for

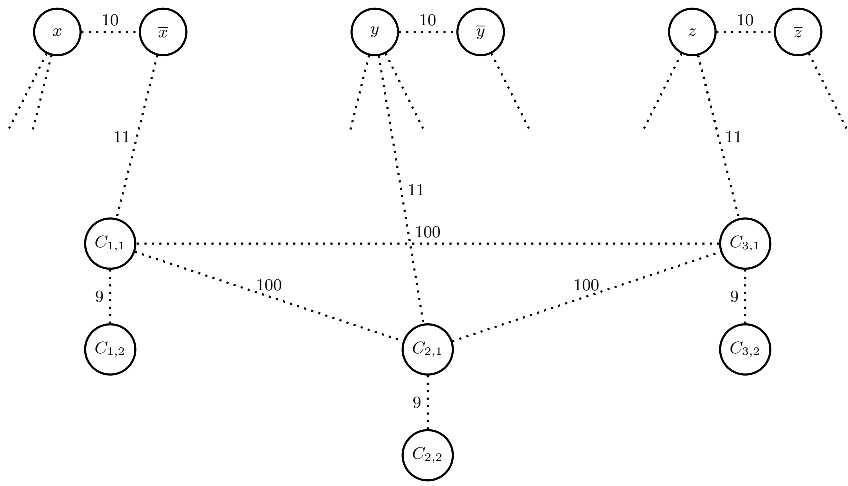


Fig. 1. Gadget for $C = \overline{x} \vee y \vee z$

$S \in \mathcal{C}_3$ and $w(S) = 11$ for $S \in \mathcal{C}_4$. For α we choose $\frac{10}{21}$ and we want to reach a value of $W = 10n + 109m$. Note that each set $C_{j,1,1}, C_{j,2,1}, C_{j,3,1}$ allows one coalition of value 100 and this coalition cannot be destabilized by any of the smaller intersecting coalitions.

Assume that the 3SAT formula is satisfiable and consider some satisfying assignment. For each coalition in $\mathcal{C}_1$ we offer the $1 - \alpha$ share of its value to the player which represents the value of the variable in the satisfying assignment. Further every coalition in $\mathcal{C}_4$ offers α of its value to its variable player. All other coalitions are distributed equally and we chose $\mathcal{S}$ to be $\mathcal{C}_1$ as well as some $\{C_{j,i,1}, C_{j,i,2}\}$ with li_j satisfied and the coalition between the two remaining $C_{j,\cdot,1}$ players for every $j = 1 \dots k$. Then $\mathcal{S}$ has exactly value $10n + 109m$. Assume that $\mathcal{S}$ is not core-stable, that is, there is some $S \in \mathcal{C} \setminus \mathcal{S}$ such that both players would strictly prefer S. As $\mathcal{C}_1$ is fully in $\mathcal{S}$ $S \notin \mathcal{C}_1$. If S would be of $\mathcal{C}_2$ then one of the players $C_{j,\cdot,1}$ of S would be in a coalition of $\mathcal{C}_3$ and offered $50 > w(S)$ from there, so $S \notin \mathcal{C}_2$. Further $S \notin \mathcal{C}_3$ as again one of the players is in a coalition of $\mathcal{C}_3$ and thus offered the same from its coalition as from S. This leaves $S \in \mathcal{C}_4$. Now if the player v which S shares with the coalition of $\mathcal{C}_1$ is part of the satisfying assignment it is offered $(1 - \alpha)10 = \frac{110}{21} = \alpha w(S)$ and thus does not want to switch. If v is not in the satisfying assignment then player $C_{j,\cdot,1}$ of S is in a coalition of $\mathcal{C}_3$. Hence again no switch is desired.

Now assume we have a distribution which offers at least α of each coalition-value to each included player and a core-stable coalition structure $\mathcal{S}$ of size $\geq 10n + 109m$. At first we note that in every set $C_{j,1,1}, C_{j,2,1}, C_{j,3,1}$ two of the players build a coalition as $100\alpha > (1-\alpha)w(S)$ for $S \in \mathcal{C}_2 \cup \mathcal{C}_4$. The sum of these coalitions provides a value of $100m$. Thus in each of these sets one player $C_{j,\cdot,1}$ remains. Each coalition of $\mathcal{C}_4$ provides a value of 11 and there can be at most $min\{2n, m\}$ such coalitions in $\mathcal{S}$. Further h coalitions of $\mathcal{C}_4$ rule out h of the m

possible coalitions of $\mathcal{C}_2$ (after placing the $\mathcal{C}_3$-coalitions) and $\lceil \frac{h}{2} \rceil$ coalitions of $\mathcal{C}_1$. Thus the value of a coalition structure which holds h coalitions of $\mathcal{C}_4$ has a value of at most $11h + (m-h)9 + (n - \lceil \frac{h}{2} \rceil)10 + 100m = 10n + 109m + (\lfloor \frac{h}{2} \rfloor 11 - 9h) < 10n + 109m$ for $h > 0$. Then $\mathcal{S}$ must consist of all of $\mathcal{C}_1$, m coalitions of $\mathcal{C}_2$ and m coalitions of $\mathcal{C}_3$ (one of each for each clause) to meet the value-limit. We observe that $\alpha \cdot w(S) = \frac{110}{21} = (1-\alpha)w(S')$ but $\alpha \cdot w(S) = \frac{110}{21} > \frac{99}{21} = (1-\alpha)w(S'')$ for every $S \in \mathcal{C}_4$, $S' \in \mathcal{C}_1$ and $S'' \in \mathcal{C}_2$. By assumption $\mathcal{C}$ can be stabilized with a lower bound of α. Thus every coalition S of $\mathcal{C}_4$ which is not overlapping with some coalition of $\mathcal{S} \cap \mathcal{C}_3$ has to be stabilized via the overlapping S' of $\mathcal{C}_1$ and for this S' has to offer $1-\alpha$ of its value to the player of S leaving only $\alpha \cdot w(S')$ for the other player. Especially S' cannot stabilize any $\mathcal{C}_4$-coalitions which overlap with S' through the other player. Now to each variable x we assign the value of the player of $\{x, \overline{x}\}$ which gets offered the $1-\alpha$ share. If non of the players gets offered that much, we just assign some value randomly. We know that for every clause there is one coalition $S \in \mathcal{C}_4$ which is not intersecting with some edge of $\mathcal{S} \cap \mathcal{C}_3$. As $\mathcal{S}$ is stable, this coalition is stabilized by the $1-\alpha$ share of its intersecting variable-coalition, that is, the clause is fulfilled in our assignment. Hence the formula is satisfiable. □

Corollary 2. *Given $\alpha > 0$, it is* NP*-hard to decide the largest reachable social welfare value by a core-stable solution under a α-bounded distribution scheme.*

Corollary 3. *Given $W > 0$, it is* NP*-hard to decide the value of the largest α such that some α-bounded distribution scheme can stabilize at least one coalition structure of value at least W.*

Intuitively, finding the largest α gets easier when the coalition structure to be stabilized is some social optimum given in advance. Sadly, for $k > 2$ we again show NP-hardness of this problem. Conversely for $k = 2$ we will in Section 3 below provide an algorithm implementing this task in polynomial time under mild additional constrains.

Theorem 3. *Let $k \geq 3$. Given an optimal coalition structure $\mathcal{S}^*$, it is* NP*-hard to decide whether for a given α there is an α-bounded distribution scheme such that $\mathcal{S}^*$ becomes core-stable. This even holds for instances with all coalitions of size exactly k.*

Proof. For the proof we will show a reduction of 3SAT and use only coalitions of size k. Given a 3SAT formula with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$, where clause C_j contains the literals $l1_j, l2_j$ and $l3_j$, we have $N = N_1 \cup N_2$ where $N_1 = \{x_i, \overline{x}_i \mid i = 1 \ldots n\}$ and $N_2 = \{a_{x_i,h} \mid i = 1 \ldots n, h = 1 \ldots k-2\} \cup \{a_{C,h} \mid h = 1 \ldots k-3\}$ and $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}_1 = \{\{x_i, \overline{x}_i\} \cup \{a_{x_i,h} \mid h = 1 \ldots k-2\} \mid i = 1 \ldots n\}$ and $\mathcal{C}_2 = \{\{l1_j, l2_j, l3_j\} \cup \{a_{C,h} \mid h = 1 \ldots k-3\} \mid j = 1 \ldots m\}$. The coalitions of $\mathcal{C}_1$ have value 1 and the coalitions of $\mathcal{C}_2$ have value 1.5. For $\mathcal{S}$ we have $\mathcal{C}_1$ which is an optimal coalition structure as for $k > 3$ every coalition of $\mathcal{C}_2$ rules out all other coalitions of $\mathcal{C}_2$ (through the shared $a_{C,h}$) as well as 3 coalitions of $\mathcal{C}_2$ and thus is a decline and for $k = 3$ it is possible to have more than one coalition of $\mathcal{C}_2$ in some matching but for h non-overlapping coalitions of $\mathcal{C}_2$ we loose at least $\lceil \frac{3h}{2} \rceil$ coalition of $\mathcal{C}_1$. Finally we set $\alpha = \frac{2}{2k+1}$.

Assume that the 3SAT formula is satisfiable and consider some satisfying assignment. We give a distribution scheme respecting α under which $\mathcal{S}$ is stable. For each coalition in $\mathcal{C}_1$ we offer the $1-(k-1)\alpha$ share of its value to the player which represents the value of the variable in the satisfying assignment and a α share to each other player. For each coalition S in $\mathcal{C}_2$ we pick one fulfilled literal and offer only $\alpha \cdot w(S)$ to the according player. The rest of the coalition's value is shared equally among the remaining players. Then $\mathcal{S}$ is stable as for every coalition S of $\mathcal{C}_2$ there is some $S' \in \mathcal{C}_1$ such that S' offers $(1-(k-1)\alpha)w(S') = 1-(k-1)\frac{2}{2k+1} = \frac{3}{2k+1} = \alpha \cdot w(S)$.

Now assume conversely that we have some α-bounded distribution d such that $\mathcal{S}$ is stable. If $\mathcal{S}$ is stable there is at least one player v in every coalition $S \in \mathcal{C}_2$ which does not want to switch to S. Obviously this player has to be in some coalition of $\mathcal{C}$ in $\mathcal{S}$, that is, $S \notin \{a_{C,h} \mid h = 1 \dots k-3\} \mid j = 1 \dots m\}$. Thus $v \in N_1$ and we know that $\frac{3}{2k+1} = (1-(k-1)\alpha)w(S') \geq d_{S'}(v)w(S') \geq d_S(v)w(S) \geq 1.5 = \alpha\frac{3}{2k+1}$ where S' is the coalition of $\mathcal{S}$ including v. Hence $d_{S'}(v) = 1-(k-1)\alpha$ and $d_S(v) = \alpha$. Now for every $S' \in \mathcal{C}_1$ there can be at most one player receiving a $1-(k-1)\alpha$-share. We assign the value of the player $v \in N_1 \cap S'$, where S' is the coalition belonging to x_i, to x_i if v receives the $1-(k-1)\alpha$-share of S'. If no such v exists, we assign some value randomly to x_i. Then as $\mathcal{S}$ is stable for each coalition associated with a clause at least one $v \in N_1$ is picked, that is, our assignment satisfies all clauses. □

3 Stable Matchings and Acyclic Alternating Paths

At first we note that if some matching M is not inclusion maximal, that is, can be enlarged to a bigger matching by adding some edge, it cannot be stabilized for any $\alpha > 0$. In contrast, for $\alpha = 0$ it is easy to see that every coalition structure can be stabilized. Inclusion maximality can easily be tested, so we will only deal with inclusion maximal matchings from now on. The matching case is a subclass of coalition formation with $k = 2$. Hence, some properties from Section 2 translate directly:

- The PoS and (as inclusion monotone profits always hold for matchings) the PoA are bounded by 4α, and we can compute a suitable distribution scheme in polynomial time. In particular, for $\alpha = \frac{1}{4}$ we can ensure a PoS and a PoA of 1, while for $\alpha = \frac{1}{2}$ the PoS can go up to 2.
- For a given α and a given value W, it is NP-hard to decide whether there is an α-bounded distribution scheme that admits a stable matching M with $\sum_{e \in M} w(e) \geq W$.
- For a given α, it is NP-hard to decide the value of the best matching which can be stabilized by some α-bounded distribution scheme.
- For a given W, it is NP-hard to decide the value of the largest α such that some α-bounded distribution scheme can stabilize at least one matching of value at least W.

The lower bounds on the PoS in terms of α given in Proposition 2 do not extend, as they only hold for $k > 2$. However, the simple example of 4 players, a path $e_1 = \{1,2\}, e_2 = \{2,3\}, e_3 = \{3,4\}$ and profits $w(e_1) = w(e_2) = 1$ and $w(e_2) = \frac{1-\alpha+\epsilon}{\alpha}$ for some small enough $\epsilon > 0$ already gives a lower bound of $\frac{2\alpha}{1-\alpha}$ which coincides with the upper bound of 2 at the extreme point of $\alpha = \frac{1}{2}$. Obviously e_1 and e_3 both can offer $1-\alpha$ to the vertices of the inner edge, but need $\alpha\frac{1-\alpha+\epsilon}{\alpha} = 1-\alpha+\epsilon > 1-\alpha$. This leads to a PoS of $\frac{2\alpha}{1-\alpha+\epsilon} \xrightarrow{\epsilon\to 0} \frac{2\alpha}{1-\alpha}$.

For the remainder of this section, we will show improved results by restricting our attention to a subclass of matching instances, which we term *acyclic alternating*. For defining this subclass we make the following observations.

Suppose we want to stabilize some matching M. Consider an edge $e \notin M$ which has a common endpoint v with some $e' \in M$ such that $w(e) \leq w(e')$. Such edge never is a blocking pair for M if we assign only $\alpha w(e)$ of e to v, as at least $\alpha w(e') \geq \alpha w(e)$ is offered to v by e'. Hence, for all following analyses we will assume that the distribution schemes assign $\alpha w(e)$ of e to v for all edges $e \in E \setminus M$ with $w(e) \leq w(e')$ for some adjacent $e' \in M$ with $e \cap e' = \{v\}$ and not handle them explicitly anymore. Instead, we only focus on the edges $e \in E \setminus M$ for which all adjacent $e' \in M$ have $w(e') < w(e)$. We call such edges *dominating* and their adjacent matching edges *dominated*. We denote the subgraph of G consisting of the dominating and dominated edges by $G_d(M)$ and the set of these edges by $E_d(M)$. If every path in $G_d(M)$ which alternates between M and $E \setminus M$ is acyclic, we call $G_d(M)$ *acyclic alternating*. We note that for optimal M, $G_d(M)$ cannot contain any even cycles alternating between M and $E \setminus M$ as this would contradict the optimality of M. An acyclic alternating $G_d(M)$ resulting from some optimal matching M allows us to show improved bounds.

Let us first note that the restriction to graphs with $G_d(M)$ acyclic alternating for some or even every optimal matching M is not a drastic cutback, as it covers an interesting subclass of well-studied matching problems.

Proposition 3. *Let M^* be a maximum weight matching for a given graph G and profits w. Then we have*

$$\begin{aligned} &\{(G,w) \mid G \textit{ bipartite}\} \\ \subsetneq &\{(G,w) \mid \forall M^* : G_d(M^*) \textit{ is acyclic alternating}\} \\ \subsetneq &\{(G,w) \mid \exists M^* : G_d(M^*) \textit{ is acyclic alternating}\}. \end{aligned}$$

Proof. As for optimal matchings M^* there are no even circles in $G_d(M^*)$ and bipartite graphs have no odd circles both inclusions hold obviously.

For strictness of the first inclusion consider the graph $G = (\{1,2,3\}, \{e_1 = \{1,2\}, e_2 = \{2,3\}, e_3 = \{3,1\}\})$ with values $w(e_1) = w(e_2) = w(e_3) = 1$. While G is not bipartite for all three optimal matchings M^* (each consisting of one edge) $G_d(M^*)$ is empty and thus acyclic alternating. For the second inclusion consider $G = (\{1,2,3,4\}, \{\{1,2\},\{2,3\},\{2,4\},\{3,4\}\})$ with values $w(\{1,2\}) = w(\{3,4\}) = 1$ and $w(\{2,3\}) = w(\{3,4\}) = 2$. Then for $M_1^* = \{\{1,2\},\{3,4\})$ $G_d(M_1^*)$ is not acyclic alternating while for $M_2^* = \{\{2,3\}\}$, $G_d(M_2^*)$ is. $\square$

Now we will see that the acyclic alternating property can actually help improving the lower bound on α needed for a PoS of 1:

Theorem 4. *For any optimal matching M^* such that $G_d(M^*)$ is acyclic alternating, there is an α-bounded distribution scheme that stabilizes M^* with $\alpha = \frac{1}{3}$, and this bound is tight. Given such an M^*, the distribution scheme can be computed in polynomial time.*

Proof. By definition of $G_d(M^*)$, any edge $e = \{u, v\} \in E_d \setminus M^*$ must have a matching edge on both sides, that is, there are $e_1, e_2 \in M^*$ with $u \in e_1$ and $v \in e_2$ as otherwise $M^* \cup \{e\} \setminus \{e' \mid e \cap e' \neq \emptyset\}$ would actually improve social welfare in contradiction to M^* being optimal. Thus, every inclusion maximal alternating path in $G_d(M^*)$ begins and ends with an edge of M^*. We call such an alternating path in $G_d(M^*)$ *dominated.*

In this proof we will give a distribution scheme such that for every edge $e \in M$ we have $d.(e) \in \{1/3, 2/3\}$. For convenience, we will refer to an edge $\{u, v\}$ being *oriented from u to v* if and only if u receives a share of $\alpha = 1/3$ and v receives $1 - \alpha = 2/3$.

First we will analyze a single path and then explain how to apply the resulting possible distributions to multiple overlapping paths. We start with proving a useful property for dominated paths. We call a path $P = e_1 e_2 \dots e_m$ of edges *conflicting* if $e_1, e_m \in M^*$, $w(e_1) < \frac{1}{2} w(e_2)$, $w(e_m) < \frac{1}{2} w(e_{m-1})$, and $w(e_{2i-1}) \geq \frac{1}{2} w(e_{2i}) \leq w(e_{2i+1})$ for all $i = 2, \dots, (\frac{m-1}{2} - 1)$.

Proposition 4. *There is no dominated path P in $G_d(M^*)$ which contains a conflicting subpath.*

Proof. Assume P conversely that a conflicting subpath $P' = e_1 e_2 \dots e_m$ exists. Now, on the one hand we know that

1. $w(e_2) - w(e_1) > w(e_1)$,
2. $w(e_{2i+2}) - w(e_{2i+1}) \geq 0$ for $i = 1, \dots, (\frac{m-1}{2} - 1)$, and
3. $\sum_{i=0}^{\frac{m-1}{2}} w(e_{2i+1}) - \sum_{i=1}^{\frac{m-1}{2}} w(e_{2i}) \geq 0$ (because M^* is optimal),

which provides

$$\begin{aligned} w(e_m) &\overset{3}{\geq} \sum_{i=0}^{\frac{m-1}{2}-1} w(e_{2i+1}) - \sum_{i=1}^{\frac{m-1}{2}} w(e_{2i}) \\ &= w(e_2) - w(e_1) + \sum_{i=1}^{\frac{m-1}{2}-1} w(e_{2i+2}) - w(e_{2i+1}) \\ &\overset{1,2}{>} w(e_1). \end{aligned}$$

On the other hand we also have

1. $w(e_{m-1}) - w(e_m) > w(e_m)$,

2. $w(e_{2i}) - w(e_{2i+1}) \geq 0$ for $i = 1, \ldots, (\frac{m-1}{2} - 1)$, and again
3. $\sum_{i=0}^{\frac{m-1}{2}} w(e_{2i+1}) - \sum_{i=1}^{\frac{m-1}{2}} w(e_{2i}) \geq 0$,

which provides

$$\begin{aligned} w(e_1) &\overset{3}{\geq} \sum_{i=1}^{\frac{m-1}{2}} w(e_{2i+1}) - \sum_{i=1}^{\frac{m-1}{2}} w(e_{2i}) \\ &= w(e_{m-1}) - w(e_m) + \sum_{i=0}^{\frac{m-1}{2}-2} w(e_{2i+2}) - w(e_{2i+1}) \\ &\overset{1,2}{>} w(e_m). \end{aligned}$$

Thus, we have $w(e_m) > w(e_1) > w(e_m)$, that is, no subpath P' with the named properties can exist in P. □

We will now show how to find a distribution scheme for a single dominated path P such that $M^* \cap P$ is core-stable regarding P. Let $P = v_1 v_2 \ldots v_m = e_1 e_2 \ldots e_{m-1}$, $v_1 \ldots v_m \in V$, $e_1 \ldots e_{m-1} \in E$, be a dominated path. Then there is some j_r and some j_l such that $w(e_{2i-1}) \geq \frac{1}{2} w(e_{2i})$ for $i < j_r$ and either $w(e_{2j_r-1}) < \frac{1}{2} w(e_{2j_r})$ or $2j_r > m$, and $w(e_{2i+1}) \geq \frac{1}{2} w(e_{2i})$ for $i > j_l$ and either $w(e_{2j_l+1}) < \frac{1}{2} w(e_{2j_l})$ or $j_l = 0$. We call j_l and j_r the *bounds of* P.

We claim that $j_l \leq j_r$. Assume conversely that $j_r < j_l$. Then P holds the subpath $P' = e_{2j_r-1} \ldots e_{2j_2+1}$ with $w(e_{2j_r-1}) < \frac{1}{2} w(e_{2j_r})$, $w(e_{2j_2+1}) < \frac{1}{2} w(e_{2j_2})$, and $w(e_{2i-1}) \geq \frac{1}{2} w(e_{2i}) \leq w(e_{2i+1})$ for all $i = j_r + 1 \ldots j_l - 1$. By Proposition 4 such a subpath cannot exist in P. Thus $j_l \leq j_r$ and the bounds describe a feasible interval $e_{2j_l} \ldots e_{2j_r}$ of P. We call this interval the *variable part of* P.

For every $2j_l \leq k \leq 2j_r$, consider the distribution scheme $d_{P,k}$ which shares the edges of P such that e_i is oriented form v_i to v_{i+1} for $i < k$ and e_i from v_{i+1} to v_i for $i \geq k$. $d_{P,k}$ is a scheme with "inward pointing" orientation which switches orientation at position k. We will show that every such distribution scheme stabilizes $P \cap M^*$ regarding P. Consider such a scheme and some $e \in P \setminus M^*$. If $e = e_i$ with $i < k$, v_i is offered an α-share of $w(e_i)$ but an $(1-\alpha)$-share from $w(e_{i-1})$. As $i < k \leq 2j_r$ we know that $w(e_{i-1}) \geq \frac{1}{2} w(e_i)$, that is, $(1-\alpha) w(e_{i-1}) = \frac{2}{3} w(e_{i-1}) \geq \frac{1}{3} w(e_i) = \alpha w(e_i)$. Thus, there is no incentive for v_i to switch from $e_{i-1} \in M^*$ to $e_i = e$ and e is non-blocking. Similarly, if $e = e_i$ with $i \geq k$, v_{i+1} is offered an α-share of $w(e_i)$ and an $(1-\alpha)$-share from $w(e_{i+1})$. As $i \geq k \geq 2j_l$ we know that $w(e_{i+1}) \geq \frac{1}{2} w(e_i)$, that is, $(1-\alpha) w(e_{i+1}) = \frac{2}{3} w(e_{i+1}) \geq \frac{1}{3} w(e_i) = \alpha w(e_i)$. Thus, there is no incentive for v_i to switch from $e_{i+1} \in M^*$ to $e_i = e$.

Subsequently, we will analyze Algorithm 3 and prove that it provides a 1/3-bounded distribution scheme for all edges of G which stabilizes M^*. Observe that this algorithm does not necessarily run in polynomial time. It merely serves as a formalization of the existence argument. For computing distribution schemes with maximal α in polynomial time, we refer the reader to Theorem 5 below.

Algorithm 3: Distribution scheme using only $\frac{1}{3} \setminus \frac{2}{3}$ shares

Data: Instance (G, w), social optimum M^*

1 set $F = E$;
2 **foreach** $e \in E \setminus M^*$ **do**
3   **if** $\exists e' \in M^*$: $w(e) \leq w(e')$ *and* $e \cap e' = \{u\}$ **then**
4   orient e away from u;
5   set $F = F \setminus \{e\}$;
6 **foreach** $P = v_1 v_2 \dots v_m = e_1 e_2 \dots e_{m-1}$ *dominated path in* $G_d(M^*)$ **do**
7   find the bounds j_l and j_r for P;
8   **foreach** $i < 2j_l$ **do**
9   orient e_i from v_i to v_{i+1};
10   set $F = F \setminus \{e_i\}$;
11   **foreach** $i > 2j_r$ **do**
12   orient e_i from v_{i+1} to v_i;
13   set $F = F \setminus \{e_i\}$;
14 **while** $F \neq \emptyset$ **do**
15   choose $e = \{u, v\} \in F$;
16   orient e from u to v;
17   call **Propagate**(e, u, F);
18   call **Propagate**(e, v, F);

The algorithm works as follows. At first, in the loop starting at Line 2, we take care of all edges in $E \setminus E_d(M^*)$. Each of those edges shares an endpoint with a matching edge of equal or larger value. We can easily ensure non-blocking status by orienting the edge away from one endpoint incident to a higher-valued matching edge. Next, in the loop starting at Line 6, for every dominated path we orient edges outside the variable part using "inward pointing" as above. We need to show that for each edge e, although we might handle it repeatedly due to overlapping dominated paths, all these paths result in the same orientation. Assume for contradiction that there are two dominated paths $P = v_1 \dots v_m = e_1 \dots e_{m-1}$ and $P' = v'_1 \dots v'_{m'} = e'_1 \dots e'_{m'-1}$ and an edge $e = \{u, v\}$ such that P induces an orientation of e from u to v while P induces an orientation from v to u. Let j_l and j_r be the the bounds of P and j'_l and j'_r the bounds of P'. Then $e = e_i$ with $i < 2j_l$ or $i > 2j_r$ and $e = e'_{i'}$ with $i' < 2j'_l$ or $i > 2j'_r$. We only discuss the case that $i < 2j_l$ and $i' < 2j'_l$ as all other cases are similar. Consider the path $P'' = e_{m-1}e_{m-2} \dots e_i e_{i'+1} \dots e'_{m'}$ and note that $e_i = e_{i'}$. The fact that $i < 2j_l$, $i' < 2j'_l$, and the orientations given by P and P' are contradictory tells us that in P we have $u = v_i$ and $v = v_{i+1}$ while in P' we have $u = v'_{i'+1}$ and $v = v'_{i'}$. Thus P'' is again a dominated path. Then for P'' again we have bounds j''_l and j''_r. Now applying the definition to P'' it turns out that the position of $2j''_r$ coincides with the position of $2j_l$ in the P-part and the position of $2j''_l$ coincides with the position of $2j'_l$ in P'. But then $j''_r < j''_l$ which was already proven to be impossible.

Further, let us observe that once the algorithm reaches Line 14, there is no dominated path $P = v_1 \dots v_m = e_1 \dots e_{m-1}$ and indices $i_1 < i_2 < i_3$ such

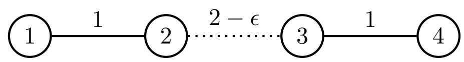
Algorithm 4: Propagate

```
Data: edge e, vertex u, set of undetermined edges F
if e ∈ M* then
    foreach e' = {u, v'} ∈ F \ M* do
        orient e' from v' to u;
        set F = F \ {e'};
        call Propagate(e', v', F);
else
    foreach e' = {u, v'} ∈ F ∩ M* do
        orient e' from v' to u;
        set F = F \ {e'};
        call Propagate(e', v', F);
```

that e_{i_1} and e_{i_3} are oriented in the same direction (either e_{i_1} from v_{i_1} to v_{i_1+1} and e_{i_3} from v_{i_3} to v_{i_3+1} or e_{i_1} from v_{i_1+1} to v_{i_1} and e_{i_3} from v_{i_3+1} to v_{i_3}) but $e_{i_2} \in F$. Put differently, e_{i_2} has no orientation yet, or e_{i_2} is oriented in the opposite direction. Assume for contradiction that there is such path P and indices $i_1 < i_2 < i_3$. We will consider the case that e_{i_1} is oriented from v_{i_1} to v_{i_1+1} and e_{i_3} from v_{i_3} to v_{i_3+1}. The other case works similarly. As e_{i_3} was not oriented when P was considered, there must be some path $P' = v'_1 \dots v'_{m'} = e'_1 \dots e'_{m'-1}$ which shares e_{i_3} with P, say $e_{i_3} = \{v_{i_3}, v_{i_3+1}\} = \{v'_{i'}, v'_{i'+1}\} = e'_{i'}$ (assuming that the vertices in P' are labeled such that P and P' are both increasing in the same direction on e_{i_3}). Let j_l and j_r denote the bounds of P and j'_l and j'_r denote the bounds of P'. As e_{i_2} is not oriented by P, we know that $2j_l < i_2 < 2j_r$ and, as e_{i_3} is oriented from $v'_{i'}$ to $v'_{i'+1}$, we know that $i' < j'_l$. Consider $P'' = v_1 \dots v_{i_2} v'_{i'+1} \dots v'_{m'}$, where $v_{i_2} = v'_{i'}$. Now P'' again is a dominated path and further the bound j''_l of P'' coincides with j'_l of P'. But P'' was considered in Line 6 as well and caused the orientation of all edges outside of its variable part between j''_l and j''_r. In particular, P'' caused the orientation of e_{i_2}, and e_{i_2} is oriented in the same direction as e_{i_1} and e_{i_3}. Note, that this means that at Line 14 for every dominated path P the part $P \cap F$ of unoriented edges is connected.

Now we come to the final part starting at Line 14, where we orient the intersection of variable parts of all paths. Intuitively, in this part we can simply pick a random edge, which is not oriented yet, and choose its v-vertex as the position where the switch of orientation should take place. Then, using the Propagate subroutine outlined in Algorithm4 we ensure that for every dominated path which includes v the remaining unoriented edges get consistently oriented (towards v). Thus for each dominated path we obtain a distribution scheme as discussed above, that is, the non-matching edges in each path are non-blocking. Together with the edges in $G \setminus G_d(M^*)$ that were already handled in the first loop, we thus have ensured that all non-matching edges are non-blocking. Hence M^* is stable.

To show tightness, we consider the following simple path, in which any value $\alpha > \frac{1}{3}$ will fail to stabilize the unique M^*.

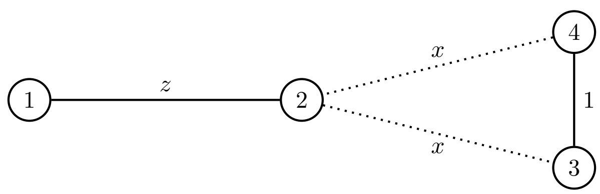


Let $\alpha > \frac{1}{3}$ and $0 < \epsilon < \frac{3\alpha-1}{\alpha}$. Then, the unique M^* consisting of $\{1,2\}$ and $\{3,4\}$ cannot be stabilized as $1-\alpha = 2\alpha-(3\alpha-1) = \alpha(2-\frac{3\alpha-1}{\alpha}) < \alpha(2-\epsilon)$. □

Note that this bound gives a real improvement compared to graphs without acyclic alternating structure:

Proposition 5. *There are matching games where a PoS of 1 requires* $\alpha = 1-\sqrt{\frac{1}{2}} \approx 0.2929 < \frac{1}{3}$.

Proof.



We consider the example above resulting in an upper bound of $\alpha = 1-\sqrt{\frac{1}{2}} \approx 0.2929$. Let $N = \{1,2,3,4\}$, $\mathcal{C} = \{\{1,2\},\{2,3\},\{2,4\},\{3,4\}$ with $w(\{1,2\}) = z$, $w(\{2,3\}) = w(\{2,4\}) = x$ and $w(\{3,4\}) = 1$. In this small example assuming that $\mathcal{S} = \{\{1,2\},\{3,4\}\}$ is the social optimum the best strategy on $\{3,4\}$ is obliviously to give $\frac{1}{2}$ to each side. Let α be the lower bound on the share each player should at least get. Then $\{1,2\}$ will be shared such that 1 gets αz and 2 gets $(1-\alpha)z$ as 1 as no option to switch while 2 might get tempted by the x coalitions. We know that $1+z > x$ and further need either $\frac{1}{2} \geq \alpha x$ or $(1-\alpha)z \geq \alpha x$. The most extreme α we can derive for these constrains is $\alpha = 1-\sqrt{\frac{1}{2}}$ for $x = \frac{1}{2-2\sqrt{\frac{1}{2}}}$ and $z = \frac{1}{2\sqrt{\frac{1}{2}}}$. □

In Section 2 we observed NP-hardness of deciding whether a given α suffices to stabilize some optimal matching. Using the acyclic alternating property, we can optimize α to stabilize arbitrary matchings. Hence, this property helps not only in stabilizing optimal but any given inclusion maximal matching with optimal α.

Theorem 5. *Given a matching M such that $G_d(M)$ is acyclic alternating and some $\alpha \in \left[0,\frac{1}{2}\right]$, we can decide in polynomial time if there is an α-bounded distribution scheme stabilizing M.*

Algorithm 5: Computing a Distribution Scheme for M

Data: Instance (G, w), matching M, bound α

set $S = E$;
foreach $e \in E \setminus M$ **do**
  if $\exists e' \in M : e \cap e' = \{u\}$ *and* $w(e) \leq w(e')$ **then**
    share e such that $\alpha w(e)$ is offered to u;
    set $S = S \setminus \{e\}$;
foreach $e = \{u, v\} \in M$ **do**
  set $s_u = \alpha w(e)$, $s_v = \alpha w(e)$ and $rest_e = (1 - 2\alpha) w(e)$;
while $S \neq \emptyset$ **do**
  if $\exists e = \{u, v\} \in M : u \notin e' \forall e' \in S \setminus \{e\}$ **then**
    set $s_v = s_v + rest_e$ and $rest_e = 0$;
    share e such that s_u is offered to u and s_v is offered to v;
    set $S = S \setminus \{e\}$;
    foreach $e' \in S$ *with* $e' \cap e = \{v\}$ **do**
      if $s_v \geq \alpha w(e')$ **then**
        share e' such that $\alpha w(e')$ is offered to v;
        set $S = S \setminus \{e'\}$;
  if $\exists e = \{u, v\} \in E \setminus M : u \notin e' \forall e' \in S \cap M$ **then**
    if $\exists e' \in S \cap M : v \in e'$ *and* $s_v + rest_{e'} \geq \alpha w(e)$ **then**
      if $s_v < \alpha w(e)$ **then**
        set $s_v = \alpha w(e)$ and $rest_{e'} = rest_{e'} - (\alpha w(e) - s_v)$;
      share e such that $\alpha w(e)$ is offered to v;
      set $S = S \setminus \{e\}$;
    else
      return 'M cannot be stabilized with lower bound α';

Proof. We have already seen how to find a distribution scheme for $\alpha \leq \frac{1}{3}$ in Theorem 4. Here we will treat the slightly more general approach as shown in Algorithm 5. We first describe the intuitive idea behind the algorithm. The main idea of the algorithm for $G_d(M)$ is using that in every round there is an edge for which the profit of one agent is already determined by the algorithm. In particular, for $e \in M$ there are only edges $e' \notin M$ on one side that we have to make non-blocking and for $e \in E_d(M) \setminus M$ there is only one $e' \in M$ left on one side which can be used to make e non-blocking. This property is due to the fact that no alternating path in $G_d(M)$ contains a cycle. Deciding the distribution for $e \in E_d(M) \setminus M$ is easy. We give the smallest possible value $\alpha w(e)$ to the side where an edge $e' \in M$ is supposed to ensure non-blocking status of e. In contrast, for matching edges $e \in M$ we have to be more careful. We start by giving each side only the minimal portion of $\alpha w(e)$ and keep the rest as buffer. Now every time we encounter an $e' \in E_d(M) \setminus M$ which can only be stabilized by e, we raise the share on that side just as much as needed using up some of the buffer. If such e' are left only on one side, we can push all the remaining buffer to that side and check whether (some of) the e' become non-blocking by this assignment.

More formally consider the execution of Algorithm 5. At first we show that we will not get stuck in the while-loop, that is, in every execution of the loop at least one edge is removed from S (or we find out that M cannot be stabilized and stop). Assume conversely that although $S \neq \emptyset$, there neither is a matching edge with no adjacent edges at one side nor a non matching edge with no adjacent matching edge at one side. Then S contains a cycle which alternates between matching and non matching edges in contradiction to the properties of $G_d(M)$.

Next we will see that, if the algorithm does not terminate early, the final output is an α-bounded distribution scheme which stabilizes M. Every edge e of $E \setminus M$ is shared $(\alpha w(e), (1-\alpha)w(e))$ and every edge $e = \{u, v\}$ in M is shared according to (s_u, s_v) which is a valid distribution as throughout the algorithm $s_u + s_v + rest_e = w(e)$, where the buffer $rest_e = 0$ in the end. Obviously we never exceed nor undercut the value of an edge. Further, s_u and s_v start at $\alpha w(e)$ and increase monotonically. Thus, our distribution respects the lower bound of α. We claim that every $e \in E_d(M) \setminus M$ that is dropped from S is already stabilized by an offer according to the current value s_u of some incident agent u. Then, with S empty in the end and all matching edges shared according to the s_u-values (that increase over the run of the algorithm) M is indeed stable. If e has been removed at Line 5, the agent (denoted u) to which $\alpha w(e)$ is offered will get at least as much from its incident matching edge e'. If e has been removed at Line 16, it is stabilized by the matching edge which is removed along with it. Further, each edge removed at Line 22 is stabilized as well, as the s_v-value is adjusted to be large enough if it has not been before. Hence M is stable under the given distribution.

Now assume that the algorithm terminates early declaring that M cannot be stabilized. Obviously, at the point where this decision is made, the currently examined edge $e \in E_d(M) \setminus M$ cannot be made non-blocking anymore, because either the incident matching edges where fully shared and removed already without the share being large enough to make e non-blocking and delete it from S (Line 16) or the current offer combined with the remaining buffer of the incident matching edge is smaller than $\alpha w(e)$. Thus, we have to show that we did not offer a bigger share than needed to the other side earlier. In the beginning each s_v is set to $\alpha w(e)$ where e is the matching edge containing v, so nothing is wasted. Now there are only two points where s_v is enlarged. If s_v is changed in Line 10, we already know there are only edges on the side of v which remain to be stabilized, as we have seen above that every dropped edge is already stabilized. Thus by giving the rest of the buffer to v we waste nothing for the other side. The other time s_v is changed is at Line 20 where it is enlarged to meet the minimal possible offer of the currently examined non matching edge $e = \{u, v\}$. Now e has been picked because there is no matching edge left on one side of it, that is, either u is not matched in M, then e has to be stabilized at v and thus it is necessary to rise s_v, or the matching edge of u is already deleted from S. But when deleting a matching edge we always ensure to delete all non matching edges which get stabilized by the matching edge as well. Hence, again it is necessary to stabilize e at v, and the algorithm only terminates early, if it is not possible to stabilize M while respecting the α-bound. Together with the fact that the algorithm provides an α-bounded distribution scheme under which M is stable, if it terminates with $S = \emptyset$, this proves the theorem. □

Proposition 6. *Suppose we are given a matching M and an α-bounded distribution scheme with maximal α stabilizing M. There are at most $|E|^3$ many relevant values for such a maximal α, each computable in polynomial time. This holds even if $G_d(M)$ is not acyclic alternating.*

Proof. Consider some edge $e = \{u, v\} \in M$ and let $L_e = \{e' | u \in e' \in E \setminus M, w(e') > w(e)\} \cup \{e_a\}$ and $R_e = \{e' | v \in e' \in E \setminus M, w(e') > w(e)\} \cup \{e_a\}$ where we assume e_a to be some auxiliary edge of value $w(e_a) = w(e)$. Now let $e_1 \in L_e$ be the edge of highest value in L_e which has to be put to non-blocking status by e and, similarly, $e_2 \in R_e$ be the edge of highest value in R_e which has to be put to non-blocking status by e. If on some side there are no edges which have to be handled, we choose e_a to ensure an offer of α. Now the largest α which allows e putting both e_1 and e_2 (and all smaller edges on the respective sides) to non-blocking fulfills exactly $w(e) = \alpha w(e_1) + \alpha w(e_2)$. As $|M|$, $|L_e|$ and $|R_e|$ are all of size at most $|E|$, the number of different such α-values arising from M is limited by $|E|^3$. We claim that the maximum α for an α-bounded distribution scheme stabilizing M must be among these candidate values. Assume conversely the optimal α^* does not fulfill the equation $w(e) = \alpha^* w(e_1) + \alpha^* w(e_2)$ for any $e \in M$, $e_1 \in L_e$, $e_2 \in R_e$. Now consider some α^*-bounded distribution scheme d which stabilizes M. For each $e \in M$ let e_1^* be the worthiest edge of L_e which is non-blocking and e_2^* the worthiest edge of R_e which is non-blocking because of e under d. We know that $\alpha^* w(e_1^*) + \alpha^* w(e_2^*) \lneq w(e)$ for every $e \in M$. Let $\alpha^+ = \min\{\alpha \mid \alpha w(e_1^*) + \alpha w(e_2^*) = w(e) \text{ for some } e \in M\}$. Then $\alpha^+ > \alpha^*$ and we can stabilize M with an α^+-bounded distribution scheme in the following way. We share each edge e' in $E \setminus M$ such that α^+ is offered to (one of) the matching edge which ensures non-blocking status for e' in d, and we share each $e \in M$ such that $d_u(e)w(e) \geq \alpha^+ w(e_1^*)$ and $d_v(e)w(e) \geq \alpha^+ w(e_2^*)$ for its respective e_1^* and e_2^*. This contradicts maximality of α^* and completes the proof. □

Corollary 4. *Given a matching M such that $G_d(M)$ is acyclic alternating, we can in polynomial time find the maximal bound α for which M can be stabilized as well as suiting distribution scheme.*

Observe that for general matching games, the relevant α-values can be bounded and computed in the same way, even if $G_d(M)$ is not acyclic alternating. However, in general it is not clear how to use this information to construct an optimal distribution scheme, as it remains to decide which matching edges have to stabilize which non-matching edges within cycles.

The characterization for the number of candidate values for optimal α can be directly generalized to larger coalitions using the same arguments. However, we have already seen in Theorem 3 that even the knowledge of the optimal value for α does not help in finding a stabilizing distribution scheme efficiently.

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