On the Complexity of Pareto-optimal Nash and Strong Equilibria *

Martin Hoefer[†] Alexander Skopalik[‡]

Abstract

We consider the computational complexity of coalitional solution concepts in scenarios related to load balancing such as anonymous and congestion games. In congestion games, Pareto-optimal Nash and strong equilibria, which are resilient to coalitional deviations, have recently been shown to yield significantly smaller inefficiency. Unfortunately, we show that several problems regarding existence, recognition, and computation of these concepts are hard, even in seemingly special classes of games. In anonymous games with constant number of strategies, we can efficiently recognize a state as Pareto-optimal Nash or strong equilibrium, but deciding existence for a game remains hard. In the case of player-specific singleton congestion games, we show that recognition and computation of both concepts can be done efficiently. In addition, in these games there are always short sequences of coalitional improvement moves to Pareto-optimal Nash and strong equilibria that can be computed efficiently.

1 Introduction

A central theme of (algorithmic) game theory is the study and analysis of equilibria to predict the outcomes of interacting rational agents. Insights about the nature of equilibria yield numerous benefits, e.g., for the design and implementation of regulations such as laws in society or protocols in distributed systems. In strategic games the most frequently studied concept of stability is the Nash equilibrium (NE) – a state, in which no agent has an incentive to unilaterally deviate. The analysis of Nash equilibrium has occupied a central place in game theory since its beginning. More recently, the computational complexity of Nash equilibrium has been analyzed to determine whether the concept is reasonable from a computational point of view [1, 11].

Much of the attractiveness of Nash equilibrium stems from its elegance and simplicity and (in the mixed case) from guaranteed existence. However, Nash equilibrium is only resilient against *unilateral* deviations. It neglects the aspect of *cooperation* or *coordination* between agents. Obviously, in many scenarios agents have an incentive to cooperate, as cooperation often allows to dramatically improve the situation of every participant. In these cases, the negligence of cooperation in Nash equilibrium significantly hurts the explanatory power and predictive value of the concept in practice.

This shortcoming of Nash equilibrium has been addressed already in the 1950s, most notably by Aumann [3] who introduced the strong equilibrium (SE) – a state, from which no coalition

*A preliminary version of this paper appeared in the proceedings of the 3rd International Symposium on Algorithmic Game Theory (SAGT 2010) [16].

†Supported by DFG through UMIC Research Center at RWTH Aachen University and grant Ho 3831/3-1. Department of Computer Science, RWTH Aachen University, Germany. mhoefer@cs.rwth-aachen.de.

‡Supported in part by the German Israeli Foundation (GIF) under contract 877/05. Department of Computer Science, RWTH Aachen University, Germany. skopalik@cs.rwth-aachen.de.

of agents can jointly deviate and thereby strictly improve all members of the coalition. Strong equilibria include the consideration of cooperation, but this comes at the expense of guaranteed existence. Hence, using strong equilibria we can make better predictions about the outcome in many but not all games. In addition, strong equilibria have recently been shown to exhibit a significantly smaller inefficiency in congestion and load balancing games [2, 7, 12]. Similar results have been obtained for a weaker concept of Pareto-optimal Nash equilibria (PoNE) [19], in which only unilateral deviations or deviations of the whole player set are allowed. From a designer perspective, it thus appears attractive to design (distributed) algorithms for cooperation between agents that allow to reach these states if they exist. The analysis of the computational complexity of SE and PoNE has been posed as an open problem in [7] and is the subject of this paper.

Related Work and New Results

In this paper, we examine the computational complexity of SE and PoNE in games related to congestion and load balancing. In particular, we consider problems of the following types. *Existence:* Does a given game have a SE? *Recognition:* Is a given state of a game a SE? *Computation:* If a game has a SE, can we compute it in polynomial time? We consider these problems for SE and PoNE and other related variants. In general, our results shed light on the inherent complexity of cooperation. While in some cases, we can give efficient algorithms, most of our insights turn out to be hardness results.

In Section 3 we study anonymous games [4, 5], in which the cost of a player does not depend on the identity of the other players. A notable case are games with a constant number of strategies, in which the existence of pure NE can be decided efficiently [6], and for mixed NE there exists an FPTAS [8, 9]. In this case, we can decide the recognition problem efficiently for SE and PoNE. Our algorithm uses computation of perfect matchings together with careful enumeration to find a coalition and a profitable deviation if they exist. Deciding the existence problem for SE and PoNE for a given anonymous game, however, is strongly NP-complete, even for a small constant number of strategies. Note that this is in contrast to general graphical games, where the existence problem is even Σ_2^P-complete and thus at the second level of the polynomial hierarchy [13].

An important class of anonymous games are cases of load balancing, i.e., player-specific singleton congestion games [21]. Previous work has shown existence [18] for such games with non-decreasing cost functions. However, we are not aware of any result providing efficient algorithms to compute SE or PoNE. We show in Section 4 how to obtain a SE in polynomial time and how to recognize a given state as a SE or PoNE. Interestingly, our results imply that there always exist sequences of coalitional improvement moves to SE and PoNE that are of polynomial length. We show how to obtain these moves for the players efficiently.

In Section 5 we consider standard congestion games [22] with special structure. In congestion games it has been shown that SE can be absent [17], and a characterization result has been given that describes structures of strategy spaces that always allow SE for any set of non-decreasing latency functions. An extension of SE to correlated strategies has been considered in [23]. More recently, it has been shown that in a bottleneck variant of congestion games SE exist [15], and that SE in symmetric network and matroid games can be computed in polynomial time [14]. In standard matroid and symmetric network games NE can be computed efficiently [1, 11]. In addition, there is a plethora of work on the complexity of NE in standard, weighted or integer-splittable congestion games [10, 20], or local-effect games [20].

We here treat standard congestion games and aim to draw a more detailed picture beyond the characterization of [17]. Unfortunately, even when the strategy space has simultaneously a symmetric network and matroid structure, the existence problem for SE is strongly co-NP-hard. This is particularly interesting in light of the positive results in related work mentioned above. Additionally, we can even show weak NP-hardness for such games that have only 2 players. This directly implies the hardness result also for PoNE, and k-SE (in which only coalitions of size at most k are allowed), for any $k \geq 2$.

Finally, we conclude in Section 6 with some open problems and directions for further research.

2 Definitions

Strategic Games. A strategic game $\Gamma = (N, (S_i)_{i \in N}, (c_i)_{i \in N})$ has a finite set $N = \{1, \ldots, n\}$ of players. Player $i \in N$ has a set S_i of strategies. A *state* $s \in S = S_1 \times \cdots \times S_n$ is sometimes referred to as a *strategy profile* or *profile*. The *cost function* of player i is $c_i : S \to \mathbb{R}$, which maps each state $s \in S$ to a real number. We here denote by $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$. A state $s \in S$ is a *k-strong equilibrium* (k-SE) if no subset of the players $I \subseteq N$ with $|I| \leq k$ can benefit from jointly deviating from their strategies. Formally, there is no tuple $(s', I) \in S \times 2^N$ with $s' \neq s$ and $|I| \leq k$ such that $\forall i \in I$ we have $c_i(s') < c_i(s)$ and $\forall i \in N \setminus I$ it holds $s_i = s'_i$. A n-SE is called *strong equilibrium (SE)*, and a 1-SE is a *Nash equilibrium (NE)*. A *Pareto-optimal Nash equilibrium (PoNE)* is a NE s, in which there is no other state s' with $c_i(s') < c_i(s)$ for every $i \in N$.

Anonymous Games. An anonymous game is a tuple $(N, E, (c_i)_{i \in N})$, where E is a set of resources and the strategy space of every player, i.e., $S_i = E$ for all $i \in N$. The cost function of player i depends only on the *numbers* of players that have chosen the strategies, but not their identities. More formally, for a state $s = (s_1, \ldots, s_n)$, we define the *load* $l_e(s)$ on resource e by $l_e(s) = |\{i \mid e = s_i\}|$, that is $l_e(s)$ is the number of players that selected resource e as strategy in s. We call the tuple $(l_e(s))_{e \in E}$ the *load profile* of s. Let L be the set of all load profiles. The cost function of player i is $c_i : E \times L \to \mathbb{R}$, which maps the strategy of player i and the load profile of s to a real number. The function of player i depends only on numbers of other players but not on their identity (i.e., which *set* of other players he shares his resource with). However, two players i and j choosing the same resource e can suffer a different cost, as c_i and c_j might map $l(s)$ to a different cost value. An interesting subclass are anonymous games with a *constant-size strategy set* in which the size of E is a fixed constant, which we study in this paper. Another subclass of anonymous games that we study are **player-specific singleton congestion games**. While we assume that these games can have an arbitrary number of strategies, the crucial adjustment is that the cost function of player i is $c_i(l_{s_i}(s)) \in \mathbb{R}$. Thus, it depends only on $l_{s_i}(s)$ of the resource chosen by player i. We assume that cost functions are non-decreasing.

Congestion Games. A congestion game is a tuple $(N, E, (S_i)_{i \in N}, (d_e)_{e \in E})$, where E is a set of resources, $S_i \subseteq 2^E$ is the strategy space of player $i \in N$, and $d_e : \mathbb{N} \to \mathbb{Z}$ is a delay function associated with resource $e \in E$. As above, we define the *load* on $e \in E$ in state s as $l_e(s) = |\{i | e \in s_i\}|$. The cost (or delay) $c_i(s)$ of player i in s is $c_i(s) = \sum_{e \in s_i} d_e(l_e(s))$. Note that symmetric congestion games can be seen as another subclass of anonymous games (albeit with a number of strategies that is possibly exponential in $|E|$), in which the cost functions have a special structure.

In terms of SE and PoNE we can also equivalently view asymmetric congestion games as anonymous games with strategy set $\bigcup_{i \in N} S_i$, where player i has a prohibitively large cost when he chooses a strategy $s_i \notin S_i$.

Several classes of congestion games are distinguished in the literature based on the combinatorial structure of the strategy spaces. In *network congestion games*, the resources are edges in a graph and the strategy space of each player is given by the set of paths between some source-sink pair of nodes [11]. If all players have the same source-sink pair, the games are termed *symmetric* network congestion games. In *matroid congestion games* the strategy space of each player is composed of the bases of a matroid [1].

3 Anonymous Games

In this section we start by considering the case of anonymous games with a constant number of strategies. In this case we can decide efficiently if a given state s is a SE.

Theorem 1. *A state s of an anonymous game with a constant number of strategies can be recognized as a Pareto-optimal Nash or a strong equilibrium in polynomial time.*

Proof. We show here that we can efficiently compute a profitable deviation if it exists. For the given state s let $l(s)$ denote the load profile of s. For a given (possibly different) load profile l we present an algorithm that checks in polynomial time if there exists a profitable joint deviation from s to a state that has load profile l. The algorithm repeatedly tries to compute a perfect matching in a bipartite graph. By running this algorithm for s and all polynomially many load profiles l, the theorem follows.

For a given state s and a given load profile l, we construct a bipartite *deviation graph* G and search for a perfect matching. The vertex set of graph $G = (A \cup B, F)$ is defined by $A = \{v_i \mid \text{for all } 1 \leq i \leq n\}$ and $B = \{v_{e,j} \mid \text{for all } e \in E \text{ and } 1 \leq j \leq l_e\}$. For each $1 \leq i \leq n$ and resource $e \in E$, we add all the edges $(v_i, v_{e,j})$ for all $1 \leq j \leq l_e$ if and only if $c_i(e, l) < c_i(s_i, l(s))$. In addition, there are edges $(v_i, v_{s_i,j})$ for every player $1 \leq i \leq n$ and his current strategy s_i, for every $1 \leq j \leq l_{s_i}$.

Note that a perfect matching in this graph yields an assignment of players to strategies. From this we can derive a new state s' by setting $s'_i = e$ iff $(v_i, v_{e,j})$ is in the matching for some $1 \leq j \leq l_e$. s' represents an improvement for all players i with $s_i \neq s'_i$. Therefore, if there is a profitable coalitional deviation from s to a state s' with $l(s) \neq l(s')$, the algorithm finds at least one such deviation. Observe, that for $l = l(s)$ the algorithm may return s itself. To check if there is a deviation to a strategy $s' \neq s$ with $l(s) = l(s')$, we run the algorithm n times with s and l as input. However, in the i-th run, we force player i to change his strategy by removing all edges $(v_i, v_{s_i,j})$. Thus, if there is a profitable deviation to a state $s' \neq s$ with $l(s) = l(s')$, then there will exist a perfect matching in at least one of the runs, and thereby we will find such a deviation. This proves the result for SE.

For Pareto-optimal Nash equilibria we first check if all unilateral deviations are unprofitable. For deviations of the complete set of players to a state with load profile l we use the above construction, but we add edges $(v_i, v_{s_i,j})$ if and only if they represent a strict improvement for player i, i.e., $c_i(s_i, l) < c_i(s_i, l(s))$. This implies that for each load profile l we only have to examine exactly one graph for a perfect matching. There is a perfect matching for some load profile l if and only if s is not a PoNE. This proves the result for PoNE. □

Player	Strategy	Load profile	Cost
X_i^b	On	$\lvert\text{On}\rvert = n$ and $\lvert\text{Off}\rvert = n$	2
$1 \le i \le n$	On	Otherwise	3
$b \in \{0, 1\}$	Off	$\lvert\text{On}\rvert = n$ and $\lvert\text{Off}\rvert = n$	1
	Off	Otherwise	3
	False	$\lvert\text{False}\rvert \in \{10j + 3 \mid$ for c_j contains literal x_i and $b = 0$	
		or c_j contains literal $\neg x_i$ and $b = 1\}$	1
	False	$\lvert\text{False}\rvert = 10m + 10i + 2$	1
	False	Otherwise	4
C_j^k	Verify		2
$1 \le j \le m$	False	$\lvert\text{False}\rvert = 10j + 3$	1
$1 \le k \le 10j$	False	Otherwise	3
V_i^k	Verify		2
$1 \le i \le n$	False	$\lvert\text{False}\rvert = 10m + 10i + 2$	1
$1 \le k \le 10i + 10m$	False	Otherwise	3
Prisoner_1, Prisoner_2	Cooperate	$\lvert\text{False}\rvert = 0$	5
	Cooperate	$\lvert\text{False}\rvert \neq 0$ and $\lvert\text{Cooperate}\rvert = 2$	2
	Cooperate	$\lvert\text{False}\rvert \neq 0$ and $\lvert\text{Cooperate}\rvert \neq 2$	4
	Defect	$\lvert\text{False}\rvert = 0$	5
	Defect	$\lvert\text{False}\rvert \neq 0$ and $\lvert\text{Defect}\rvert = 2$	3
	Defect	$\lvert\text{False}\rvert \neq 0$ and $\lvert\text{Cooperate}\rvert \neq 2$	1

Figure 1: Description of the cost functions in the game Γ_φ. Strategies that are not listed here have cost of 6 and, therefore, are never played in equilibrium.

We can decide for a given state whether it is a SE or not, which implies that the existence problem for PoNE and SE is in NP. In fact, deciding the existence of SE and PoNE is strongly NP-complete.

Theorem 2. *It is strongly* NP*-complete to decide if an anonymous game with a constant number of strategies has a Pareto-optimal Nash or strong equilibrium.*

Proof. We first prove the result for SE and present a reduction from 3SAT. Given a formula φ with the variables $x_1, \ldots, x_n$ and clauses $c_1, \ldots, c_m$, we construct an anonymous game Γ_φ with players X_i^0, X_i^1 (for $1 \le i \le n$), C_j^k (for $1 \le j \le m$ and $1 \le k \le 10j$), V_i^k (for $1 \le i \le n$ and $1 \le k \le 10i + 10m$), Prisoner_1, and Prisoner_2. The set of strategies is {On, Off, Verify, False, Wait, Cooperate, Defect}, costs are shown in Fig. 1.

If φ is satisfiable, let $b_1, \ldots, b_n$ be a satisfying assignment. The following state is a SE. For each $1 \le i \le n$, the player $X_i^{b_i}$ plays On and the player $X_i^{1-b_i}$ plays Off. All players C_j^k and V_i^k play Verify and players Prisoner_1 and Prisoner_2 play Cooperate.

We show that there is no coalition that can improve by jointly deviating to another state. The players $X_i^{1-b_i}$ are playing Off and have the minimal possible cost of 1. Thus, they cannot be part of a deviating coalition. The players Prisoner_1 and Prisoner_2 can improve only if some other players move to False. We will show, this cannot happen.

For the remaining players, i.e., $X_i^{b_i}$, C_j^k, V_i^k, the only possible profitable deviation is to deviate to False. Clearly, if there is a deviation of a subset of these players, it must result in $10j + 3$ (for

$1 \leq j \leq m$) or $10m + 10i + 2$ (for $1 \leq i \leq n$) players on False. We consider the former case. Assume there is a deviation of a coalition of some of the players that results in $10j' + 3$ many player on False. The coalition must contain the players $C_{j'}^1, \ldots, C_{j'}^{10j'}$ and the three players $X_i^{b_i}$ with x_i appearing in clause $c_{j'}$. However, let x_{i^*} be a variable that satisfies c_j with $x_{i^*} = b_{i^*}$. Player $X_{i^*}^{b_i^*}$ does not improve by deviation to False. Therefore, no such deviation can exist. Similarly, there is no deviation of a coalition that yields $10m + 10i' + 2$ (for $1 \leq i' \leq n$) players on False. This is only possible if both players $X_{i'}^0$ and $X_{i'}^1$ are on strategy On.

Now, assume φ is not satisfiable, and there is a strategy profile s that is a SE. We first show that in s no player is on False. If some player is on False, the players Prisoner_1 and Prisoner_2 play a game corresponding to the prisoners dilemma. This game does not admit a SE and implies that in s no player can be on False.

Now since s is an equilibrium, there are exactly n players X_i^b on On and exactly n players X_i^b on Off because otherwise they would have cost of 3. There is no $1 \leq i' \leq n$ with both players $X_{i'}^0$ and $X_{i'}^1$ being on strategy On. Otherwise, those two players and the players $V_{i'}^1, \ldots, V_{i'}^{10m+10i'}$ could jointly change to False and decrease their costs. Now, let $X_1^{b_1}, \ldots, X_n^{b_n}$ be the players on On. Since φ is not satisfiable, the assignment $b_1, \ldots, b_n$ implied by the players on On creates at least one clause $c_{j'}$ that is not satisfied. Let $x_{i'}, x_{i''}$, and $x_{i'''}$ be the three variables of this clause. Then, the players $X_{i'}^{b_{i'}}, X_{i''}^{b_{i''}}, X_{i'''}^{b_{i'''}}$, and the players $C_{j'}^1, \ldots, C_{j'}^{10j'}$ could jointly change to False and decrease their costs. This is a contradiction to the assumption that s is a SE and completes our reduction.

To show the same result for PoNE, we construct an anonymous game as described above and modify the cost functions as follows: If there is one or more players on False, the cost of each player not on False is 0.2 less than in the original game. If exactly two players are on Cooperate, the cost of each other player is 0.1 less than in the original game. For the case that the formula is satisfiable, it is easy to see that the SE described in the proof above still is a SE in the modified game. In particular, it is also a PoNE. On the other hand, if the formula is not satisfiable, every coalitional deviation that we described above now decreases the costs of all players. Therefore, no PoNE exists. □

Note that this implies that further restrictions on the games are necessary in order to decide existence or compute a SE or PoNE efficiently. We consider games with a constant number of player types, i.e., where each player has one out of a constant number of different cost functions.

Corollary 3. *In anonymous games with constant number of strategies that are (1) symmetric or (2) have only a constant number of different player types we can decide efficiently if Pareto-optimal Nash or strong equilibria exist and compute one efficiently if it exists.*

Note that for symmetric games the assignment of players in a load profile is irrelevant, hence we can use our algorithm from Theorem 1 above to check each of the polynomial number of profiles for being a SE or PoNE. For a constant number of player types, the number of essentially different assignments that can be derived from a single load profile is a polynomial number. Again, by enumeration and application of our algorithm we can decide existence and compute SE and PoNE efficiently.

4 Player-Specific Singleton Congestion Games

In this section we treat player-specific singleton congestion games. For games with non-decreasing cost functions it is known that SE always exist [18]. Here we provide efficient algorithms to compute a SE and decide whether a given state is a SE or PoNE. To the best of our knowledge these results have not been described in the literature before.

Theorem 4. *In player-specific singleton congestion games with non-decreasing cost functions we can in polynomial time (1) decide whether a given state is a Pareto-optimal Nash or strong equilibrium and (2) compute a strong equilibrium in polynomial time.*

Proof. Obviously, a state s that is a SE or PoNE must be a NE. Consider a NE s and the corresponding load profile $l(s)$. Because cost functions are non-decreasing, every profitable coalitional deviation must result in a state s' with the same load profile $l(s)$. In particular, if the load profile changes to $l(s') \neq l(s)$, there must be a resource e with higher load $l_e(s') > l_e(s)$. Consider a player moving to e. Any player moving to e does not make a strict improvement, because otherwise he could move there unilaterally – a contradiction to s being a NE. Hence, whenever we have a NE, there must be a SE with the same load profile, a fact that was observed in [18]. Every profitable coalitional deviation represents a circular switch of players and thereby decreases the sum of player costs.

We use our algorithm presented for anonymous games in Theorem 1 to decide for a given state s whether it is a SE or PoNE. Note that due to the arbitrary number of strategies, there is a possibly exponential number of load profiles. We make sure that s is a NE, then we only have to check one load profile – namely $l(s)$ – to verify that no coalitional deviation exists. In this way, we can efficiently check whether a state is a SE or PoNE.

For the task of computing a SE, we note that there are efficient algorithms to compute a NE in these games [21]. This allows us to obtain a NE s and load profile $l(s)$ in polynomial time. To compute a SE, we construct a bipartite deviation graph G for state s and target profile $l(s)$ as in the proof of Theorem 1. Here we also add costs to the edges, and let the cost of edge $(v_i, v_{e,j})$ be $c_i(l_e(s))$, for all $1 \leq j \leq l_e(s)$. Now consider any other state s' with $l(s)$, in which $c_i(s) = c_i(s')$ for every player i with $s_i = s'_i$ and $c_i(s) > c_i(s')$ for every i with $s_i \neq s'_i$. For every such state we can find a corresponding perfect matching in G. In particular, we construct a *minimum cost* perfect matching. This matching yields a state s' and we now argue that s' is indeed a SE.

Suppose for contradiction that there is a coalitional deviation from s' to a state s''. s is a NE and $c_i(s) \geq c_i(s') \geq c_i(s'')$ for every $i \in N$, with at least one inequality for a moving player. s'' must also have load profile $l(s)$, and a deviation from s' is a circular switch of players. This switch does not increase the cost of any player but decreases the cost of the moving players. Therefore, the assignment s'' is such that $c_i(s) = c_i(s'')$ for every player i with $s_i = s''_i$ and $c_i(s) > c_i(s'')$ for every i with $s_i \neq s''_i$. Note that s'' corresponds to a perfect matching in G, and the sum of costs $\sum_{i \in N} c_i(s'') < \sum_{i \in N} c_i(s')$. This is a contradiction to s' being derived from a minimum cost perfect matching in G. □

Interestingly, our proof shows that for every NE there is a single coalitional deviation that turns the state into a SE. Milchtaich [21] proved that from every state s there is a sequence of unilateral deviations with length at most $|E| \cdot \binom{n+1}{2}$ that leads to a NE. Our result implies that even SE can be reached via short sequences of improvement moves from every state of the game. In these sequences we only need one coalitional move which is efficiently computable. A similar result can

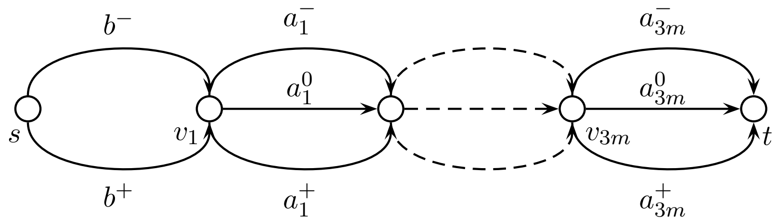

Figure 2: Construction that proves hardness of the existence and recognition problems of SE in congestion games.

be derived for PoNE, where we adjust the deviation graph to allow only coalitional improvement moves where all players strictly improve.

Corollary 5. *For every state s of a player-specific singleton congestion game with non-decreasing cost functions there is a sequence of coalitional improvement moves that leads to a strong equilibrium. Each move can be computed in polynomial time. The length of the sequence is at most $|E| \cdot \binom{n+1}{2} + 1$.*

5 Congestion Games

In this section, we consider the complexity of computing SE and PoNE in general congestion games. The class of singleton congestion games is a special case of the games we treated in the previous section, and for which we could establish a variety of positive results. Here we extend the combinatorial structure of strategy spaces only slightly to matroids. This allows to obtain a set of quite strong hardness results concerning the existence and recognition of SE and PoNE. Note that all our results in this section hold even for symmetric games, in which strategy spaces are simultaneously matroids and networks.

Theorem 6. *It is strongly* co-NP*-hard to decide (1) if a congestion game has a strong equilibrium and (2) if a given state of a game is a strong equilibrium.*

Proof. We reduce from 3-Partition. An instance is given by a multiset of integers $a_1, \ldots, a_{3m}$. Let $b = \frac{1}{m}\sum_{i=1}^{3m} a_i$. An instance $I = (a_1, \ldots, a_{3m}) \in$ 3-Partition if and only if there exists a partition of $A = \{1, \ldots, 3m\}$ into m subsets $A_1, \ldots, A_m$ such that the sum $\sum_{i \in A_j} a_i = b$ for all $1 \leq j \leq m$. Without loss of generality, we can assume that every integer $b/2 > a_i > b/4$. Therefore, each subset A_i is forced to consist of exactly three elements.

Given an instance I, we construct a congestion game Γ_I as follows. The network is $G = (V, E)$ with vertices $V = \{s, v_1, \ldots, v_{3m}, t\}$ and a series of parallel edges as depicted in Figure 2. There are $m + 1$ players. The source node of each player is s and his target node is t. The strategies are all simple s-t-paths. Hence, the game is simultaneously a symmetric network congestion game and a matroid congestion game.

The delay functions are defined as follows. Let $M = 2b$ and $1 > \epsilon > 0$. The delay of an edge a_i^- is $M - a_i$ for one player and M for more than one player. Delay of an edge a_i^+ is always $M + a_i$. The delay of an edge a_i^0 is M for at most $m - 1$ players and $2M$ for m or more players. Delay of edge b^- is $M - b - \epsilon$ for at most m players and M for more than m players. Delay of an edge b^+ is always $M + mb - \epsilon$.

If $I \in$ 3-PARTITION, we show that no SE exists. Observe that for a single agent it is never optimal to choose one of the edges a_i^+ or b^+. Thus, no SE exists in which these edges are used. Thus, in every SE every player has delay of (at least) $(3m+1)M$. However, there is a joint deviation of all players which yields delay of $(3m+1)M - \epsilon$ for each of them. Let $A_1, \ldots, A_m$ be a solution of the 3-PARTITION-instance I. Player $m+1$ choses edge b^+ and edges $a_1^-, \ldots, a_{3m}^-$. Each player $1 \leq j \leq m$ chooses edge b^- and the following edges: For each $1 \leq i \leq 3m$, if $i \in A_j$ player j plays edge a_i^+ otherwise he players a_i^0. Then the delay of a player $1 \leq i \leq m$ is

$$\begin{aligned} & (M - b - \epsilon) + \sum_{j \in A_i} (M + a_j) + \sum_{j \in \{1,\ldots,3m\} \setminus A_i} M \\ = \; & M - b - \epsilon + 3M + b + (3m-3)M \\ = \; & (3m+1)M - \epsilon \ . \end{aligned}$$

The delay of player $m+1$ is

$$\begin{aligned} & (M + mb - \epsilon) + \sum_{j=1}^{3m} (M - a_j) \\ = \; & M + mb - \epsilon + 3mM - mb \\ = \; & (3m+1)M - \epsilon \ . \end{aligned}$$

As argued above, the resulting state is not a SE either. Thus, no SE exists.

If $I \notin$ 3-PARTITION, all players choosing path $b^-, a_1^-, a_2^-, \ldots, a_{3m}^-$ is a SE. Assume for contradiction that there is a profitable deviation for a coalition. This implies that the sum of their delays also improves. If players deviate by leaving from the edges a_i^- this sum cannot decrease. Therefore, a deviation must include a deviation to the edge b^+. Without loss of generality let this be player $m+1$. Now, for player $m+1$ this deviation is only profitable if he stays on the edges $a_1^-, \ldots, a_{3m}^-$ while all other players are not on these edges. This implies for the remaining m players and each $1 \leq i \leq 3m$ that (at least) one player has to choose edge a_i^+ and (at most) $m-1$ players choose edge a_i^0. For $1 \leq j \leq m$ let $A_j = \{i \mid \text{player } j \text{ is on } a_i^+\}$. Hence, the deviation is profitable only if for all player $1 \leq j \leq m$ the following holds: $\sum_{i \in A_j} a_i \leq b$. Since we have that $\bigcup_{j=1}^m A_j = \{1, \ldots, 3m\}$, we obtain $\sum_{i \in A_j} a_i = b$ which is a contradiction to the assumption that $I \notin$ 3-PARTITION.

Obviously, the above implies that s with all players choosing path $b^-, a_1^-, a_2^-, \ldots, a_{3m}^-$ is a SE if and only if $I \notin$ 3-PARTITION. This proves co-NP-hardness of deciding whether a given state is a SE. □

Theorem 7. *It is weakly* co-NP*-hard to decide (1) if a congestion game with two players has a strong equilibrium and (2) if a given state of a game is a strong equilibrium.*

Proof. We essentially use the same construction as in Theorem 6. This time we reduce from a version of SUBSETSUM$^-$: An instance is given by a set of natural numbers $a_1, \ldots, a_m$, and b. An instance $I = (a_1, \ldots, a_m, b) \in$ SUBSETSUM$^-$ if and only if there exists a vector $(x_1, \ldots, x_m) \in \{-1, 0, 1\}^m$ with $\sum_{i=1}^m x_i a_i = b$. It is easy to show that this problem is weakly NP-complete.

Given an instance of SUBSETSUM$^-$ we construct a network congestion game as above. The network is $G = (V, E)$ with $V = \{s, v_1, \ldots, v_m, t\}$ and and a series of parallel edges as depicted in Figure 3. For the delay we use a large number $M > 3 \sum_{i=1}^m a_i$. The delay of an edge a_i is $M - a_i$

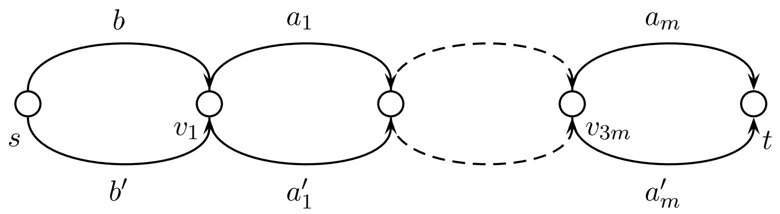

Figure 3: Construction that proves hardness of the existence and recognition problems of 2-SE and PoNE in congestion games.

for one player and M for more than one player. The delay of an edge a'_i is always $M + a_i$. The delay of edge b is $M - b - \epsilon$ for one player and M for more than one player. Delay of an edge b' is always $M + b - \epsilon$.

If $I \notin$ SUBSETSUM$^-$, the state s in which both players choose the upper path, i.e., the edges $a_1, \ldots, a_m$, and b is a SE. Note that both players have delay $(m+1)M$. Obviously, there is no improving deviation for a single player. For a joint deviation, observe that every improvement must include a deviation of one of the players to edge b'. Adequate compensation of his higher delay incurred by using this edge is only possible if the other player deviates to some of the edges a'_i. This, however, is not possible due to the assumption.

If the instance $I \in$ SUBSETSUM$^-$, consider any other state than s. Note that each player unilaterally has an incentive to switch to the upper path, hence no such state can be a NE. Now let $\{x_1, \ldots, x_m\} \in \{-1, 0, 1\}^m$ with $\sum_{i=1}^m x_i a_i = b$. The following state is a profitable deviation for both players from s. The first player chooses the edges a_i with $x_i = 0$ or $x_i = 1$ and a'_i for $x_i = -1$. He also chooses edge b'. The second player chooses the edges a_i for $x_i = 0$ or $x_i = -1$ and a'_i for $x_i = 1$. Finally, he also chooses edge b. Then the delay of the first player is

$$
\begin{aligned}
& \sum_{i:x_i=1} (M - a_i) + \sum_{i:x_i=0} M + \sum_{i:x_i=-1} (M + a_i) + M + b - \epsilon \\
= \quad & (m+1)M - \sum_{i=1}^{n} x_i a_i + b - \epsilon \\
= \quad & (m+1)M - \epsilon \ .
\end{aligned}
$$

A similar calculation shows that the delay of the second player is also $(m+1)M - \epsilon$. Finally, the above arguments imply that state s is a SE if and only if $I \notin$ SUBSETSUM$^-$. This proves the theorem. □

This implies the same result for PoNE, as for two players SE and PoNE coincide. Additionally, it implies the result for k-SE, for any $k \geq 2$.

Corollary 8. *It is weakly* co-NP*-hard to decide for a congestion game (1) if it has a k-strong equilibrium, (2) if it has a Pareto-optimal Nash equilibrium, (3) if a given state is a k-strong equilibrium, (4) if a given state is a Pareto-optimal Nash equilibrium, for any $k \geq 2$.*

6 Conclusion

In this paper, we have initiated the study of computational complexity of coalitional equilibrium concepts in anonymous and congestion games. Many of our results are hardness proofs, with some notable exceptions for player-specific singleton congestion games and recognition in anonymous games. In general, our paper opens up a variety of issues for further research. An obvious direction is to explore if there are approximate variants of SE or PoNE that exist and can be recognized and/or computed efficiently. Additionally, the social cost of such equilibria and the convergence times of natural dynamics are interesting issues for future work. More fundamentally, it is an important and challenging problem to augment the concept of pure Nash equilibrium with resilience to coalitional deviations in a meaningful way that avoids some of the devastating hardness results presented, e.g., in this paper and [13].

References

[1] Heiner Ackermann, Heiko Röglin, and Berthold Vöcking. On the impact of combinatorial structure on congestion games. *J. ACM*, 55(6), 2008.

[2] Nir Andelman, Michal Feldman, and Yishay Mansour. Strong price of anarchy. *Games Econom. Behav.*, 65(2):289–317, 2009.

[3] Robert Aumann. Acceptable points in general cooperative n-person games. In *Contributions to the Theory of Games IV*, volume 40 of *Annals of Mathematics Study*, pages 287–324. Princeton University Press, 1959.

[4] Matthias Blonski. Anonymous games with binary actions. *Games Econom. Behav.*, 28:171–180, 1999.

[5] Matthias Blonski. Characterization of pure strategy equilibria in finite anonymous games. *J. Math. Econ.*, 34(2):225–233, 2000.

[6] Felix Brandt, Felix Fischer, and Martin Holzer. Symmetries and the complexity of pure Nash equilibrium. *J. Comput. Syst. Sci.*, 75(3):163–177, 2009.

[7] Steve Chien and Alistair Sinclair. Strong and Pareto price of anarchy in congestion games. In *Proc. 36th Intl. Coll. Automata, Languages and Programming (ICALP)*, pages 279–291, 2009.

[8] Constantinos Daskalakis and Christos Papadimitriou. Computing equilibria in anonymous games. In *Proc. 48th Symp. Foundations of Computer Science (FOCS)*, pages 83–93, 2007.

[9] Constantinos Daskalakis and Christos Papadimitriou. Discretized multinomial distributions and Nash equilibria in anonymous games. In *Proc. 49th Symp. Foundations of Computer Science (FOCS)*, pages 25–34, 2008.

[10] Juliane Dunkel and Andreas Schulz. On the complexity of pure-strategy Nash equilibria in congestion and local-effect games. *Math. Oper. Res.*, 33(4):851–868, 2008.

[11] Alex Fabrikant, Christos Papadimitriou, and Kunal Talwar. The complexity of pure Nash equilibria. In *Proc. 36th Symp. Theory of Computing (STOC)*, pages 604–612, 2004.

[12] Amos Fiat, Haim Kaplan, Meital Levy, and Svetlana Olonetsky. Strong price of anarchy for machine load balancing. In *Proc. 34th Intl. Coll. Automata, Languages and Programming (ICALP)*, pages 583–594, 2007.

[13] Georg Gottlob, Gianluigi Greco, and Francesco Scarcello. Pure Nash equilibria: Hard and easy games. *J. Artif. Intell. Res.*, 24:195–220, 2005.

[14] Tobias Harks, Martin Hoefer, Max Klimm, and Alexander Skopalik. Computing pure Nash and strong equilibria in bottleneck congestion games. In *Proc. 18th European Symposium on Algorithms (ESA)*, volume 2, pages 29–38, 2010.

[15] Tobias Harks, Max Klimm, and Rolf Möhring. Strong Nash equilibria in games with the lexicographical improvement property. In *Proc. 5th Intl. Workshop Internet & Network Economics (WINE)*, pages 463–470, 2009.

[16] Martin Hoefer and Alexander Skopalik. On the complexity of Pareto-optimal Nash and strong equilibria. In *Proc. 3rd Intl. Symp. Algorithmic Game Theory (SAGT)*, pages 312–322, 2010.

[17] Ron Holzman and Nissan Law-Yone. Strong equilibrium in congestion games. *Games Econom. Behav.*, 21(1-2):85–101, 1997.

[18] Hideo Konishi, Michel Le Breton, and Shlomo Weber. Equilibria in a model with partial rivalry. *J. Econ. Theory*, 72(1):225–237, 1997.

[19] Andreu Mas-Colell, Michael Whinston, and Jerry Green. *Microeconomic Theory*. Oxford University Press, 1995.

[20] Carol Meyers. *Network Flow Problems and Congestion Games: Complexity and Approximation Results*. PhD thesis, MIT, June 2006.

[21] Igal Milchtaich. Congestion games with player-specific payoff functions. *Games Econom. Behav.*, 13(1):111–124, 1996.

[22] Robert Rosenthal. A class of games possessing pure-strategy Nash equilibria. *Int. J. Game Theory*, 2:65–67, 1973.

[23] Ola Rozenfeld and Moshe Tennenholtz. Strong and correlated strong equilibria in monotone congestion games. In *Proc. 2nd Intl. Workshop Internet & Network Economics (WINE)*, pages 74–86, 2006.