Social Context in Potential Games

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Abstract. The prevaling assumption in game theory is that all players act in a purely selfish manner, but this assumption has been repeatedly questioned by economicsts and social scientists. In this paper, we study a model to incorporate social context, i.e., the well-being of friends and enemies, into the decision making of players. We consider the impact of such other-regarding preferences in potential games, one of the most popular and central classes of games in algorithmic game theory. Our results concern the existence of pure Nash equilibria and potential functions in games with social context. The main finding is a tight characterization of the class of potential games that admit exact potential functions for any social context. In addition, we prove complexity results on finding pure Nash equilibria in numerous popular classes of potential games, such as different classes of load balancing, congestion, cost and market sharing games.

1 Introduction

Game theory deals with the mathematical study of the interaction of rational agents. A prevelant assumption in many game-theoretic works is that agents are selfish, they consider only their own well-being and act upon their own interest. The assumption that players are purely selfish disregards complicated externalities or correlations in agent interests and has been repeatedly questioned by economists and social scientists [12, 13, 19]. In many applications, agents are embedded in a social context resulting in other-regarding preferences that are not captured by standard game-theoretic models. There are numerous examples, such as bidding frenzies in auctions [22] or altruistic contribution behavior on the Internet, in which players act spiteful or altruistic and (partially) disregard their own well-being to influence the well-being of others. Despite some recent efforts, the impact of such other-regarding preferences on fundamental results in game theory is not well-understood.

In this paper, we study a general appoach to incorporate externalities in the form of other-regarding preferences into strategic games. Our model is in line with a number of recent approaches on altruistic and spiteful incentives in games. We transform a base game into another strategic game, in which players aggregate dyadic influence values combined with personal utility of other players. Relying on dyadic relations is also a popular approach in social network analysis.

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Consequently, we refer to the set of dyadic influence values as *social context* [3]. Our results concentrate on (exact) potential games, a prominent class of games with many applications that has received much attention in algorithmic game theory. Most notably, potential games always possess pure Nash equilibria, and a potential function argument shows that arbitrary better-response dynamics converge. Our interest is to understand how these conditions change when social context comes into play.

Not surprisingly, potential functions and pure Nash equilibria might not exist with social contexts, even in very simple load balancing games [3]. For a variety of prominent classes of simple potential games, such as load balancing, congestion, or fair cost-sharing games, we even show hardness of deciding existence of pure Nash equilibria. On the positive side, our main finding is a tight characterization of all games that remain exact potential games under social context. We prove that every such game is isomorphic to a congestion game with affine delays. In this sense, our characterization is similar to the celebrated isomorphism result by Monderer and Shapley [21].

Model We consider strategic games $\Gamma = (K, (\mathcal{S}_i)_{i\in K}, (c_i)_{i\in K})$ with a set K of k *players*. Each player $i \in K$ picks a *strategy* $S_i \in \mathcal{S}_i$. A *state* or *strategy profile* S is a collection of strategies, one for each player. The *(personal) cost* for player i in state S is $c_i(S)$. Each player tries to unilaterally improve his cost by optimizing his strategy choices against the choices of the other players. A state S has a unilateral *improvement move* for player $i \in K$ if there is $S_i' \in \mathcal{S}_i$ with $c_i(S_i', S_{-i}) < c_i(S)$. A state without improvement move for any player is a *pure Nash equilibrium (PNE)*.

In an *(exact) potential game*, we have a *potential function* $\Phi(S)$ such that $c_i(S) - c_i(S_i', S_{-i}) = \Phi(S) - \Phi(S_i', S_{-i})$ for every state S, player $i \in K$ and strategy $S_i' \in \mathcal{S}_i$. Φ simultaneously encodes the cost changes for all players in the game. The local optima of Φ are exactly the PNE, and every sequence of improvement moves is guaranteed to converge to such a PNE. It is well-known that every exact potential game is isomorphic to a congestion game [21]. In a *congestion game* [23] we have a set R of *resources* and for each $i \in K$ the strategy space $\mathcal{S}_i \subseteq 2^R$. For state S, we define the *load* $n_r(S)$ of resource r to be the number of players i with $r \in S_i$. Each resource $r \in R$ has a *delay* $d_r(S) = d_r(n_r(S))$, and the personal cost of player $i \in K$ is $c_i(S) = \sum_{r\in S_i} d_r(S)$.

We consider the effects of social context on the existence of potential functions. We extend a strategic game Γ by a *social context* defined by a set of weights F that contains a numerical influence value f_{ij} for each pair of players $i, j \in K$, $i \neq j$. In particular, the *perceived cost* of player $i \in K$ is given by his personal cost and a weighted average of cost of other players

$$c_i(S, F) = c_i(S) + \sum_{j\in K, j\neq i} f_{ij} c_j(S) \ .$$

We will assume throughout that the context is *symmetric*, i.e., $f_{ij} = f_{ji}$. In games with social context a state has a unilateral improvement move for player

i if i can decrease his perceived cost by switching to another strategy. A PNE in a game with social context is a state without improvement moves for perceived costs. For our lower bounds, we will restrict to *binary contexts* F with $f_{ij} = f_{ji} \in \{0,1\}$.Our existence results, however, do also allow non-binary and negative values. In the following, we say two players i and j are *friends* if $f_{ij} = f_{ji} = 1$.

We will consider social contexts in a variety of well-studied classes of potential games which we define more formally in the respective sections.

Results In Section 2 we provide the following tight characterization of the existence of potential functions in strategic games with social context. Every strategic game that admits an exact potential function for every binary context is isomorphic to a congestion game with affine delay functions. In turn, every congestion game with affine delays has an exact potential function for every social context. Hence, the class of games that allows exact potential functions for all social contexts is exactly given by congestion games with affine delays.

In the following sections we consider many popular classes of potential games and examine deciding existence of a PNE for a given game and a given binary context. In most of these games, however, a PNE might not exist and deciding existence is NP-complete. In Section 3.1 we show that this holds even for simple classes of congestion games with increasing delays, e.g., for singleton congestion games with concave delays, general congestion games with convex delays, or weighted load balancing games on identical machines. For decreasing delays, we show in Section 3.2 NP-completeness of deciding PNE existence in fair cost-sharing games, even in the broadcast case where every node of a network is a player. If we consider cost sharing with priority-based sharing rules such as the Prim rule [6], it turns out that PNE exist in undirected broadcast games, but not necessarily in directed broadcast games. However, even though equilibrium exists in undirected networks, convergence of improvement dynamics is not guaranteed. In fact, these games are not even weakly acyclic. Finally, in Section 3.3 we also briefly consider hardness of PNE existence in market sharing games. All proofs missing from this extended abstract can be found in the Appendix.

Related Work The study of social contexts and other-regarding preferences has prompted increased interest in recent years, especially in well-studied classes of potential games such as load balancing [25] or congestion games [23]. Existence of equilibrium with binary contexts and different aggregation functions in simple congestion and load balancing games was studied in [3]. Binary contexts with sum aggregation were also considered in inoculation games [20]. More recently, social cost of worst-case equilibria with and without context were quatified for general non-negative contexts in load balancing games [4]. Coalitional stability concepts in a model with social context and aggregation via minimum cost change were studied for load balancing games in [16].

Several works examined the impact of altruism on the price of anarchy [5,7,8] and equilibrium existence [17,18] in congestion and load balancing games, and in fair cost-sharing games [11]. Altruism in these works is also modelled via a weighted sum of personal and social cost. For a recent characterization of stability of social optima in several classes of games with altruism see [2].

The impact of social context with sum aggregation was also studied in other game-theoretic scenarios, for instance in auctions (see, e.g., [22] or [9] and the references therein), market equilibria [10], stable matching [1], and others.

Characterizing the existence of potential functions and pure Nash equilibria was recently discussed in weighted potential games [14, 15]. The results imply existence only for the classes of linear and exponential delay functions. This characterization refers to existence of a property for *all games from a class* with the same delay functions. In contrast, we provide a stronger result similar to [21] in the form of a one-to-one correspondence for each individual game under consideration.

2 Characterization

We start by characterizing the prevalence of potential functions under social contexts. We say a potential game has a *context-potential* Φ if there exists a function $\Phi(S,F)$ with $c_i(S,F) - c_i(S'_i, S_{-i}, F) = \Phi(S,F) - \Phi(S'_i, S_{-i}, F)$ for all states S, social contexts F, players $i \in K$, and strategies $S'_i \in \mathcal{S}_i$. Thus, a context-potential ensures that the game is potential game for every social context F. We show the following theorem.

Theorem 1. *A strategic game has a context-potential if and only if it is isomorphic to a congestion game with affine delay functions.*

We prove the theorem in two steps. We first show that a game Γ that has a context-potential for every binary context must be isomorphic to a congestion game with affine delays by constructing an isomorphic game. Afterwards, we show that these games admit a potential also for every non-binary social context by providing a context-potential.

Lemma 1. *If a strategic game has a context-potential for every binary context, then it is isomorphic to a congestion game with affine delay functions.*

Proof. It is insightful to consider an arbitrary 4-tuple of states involving the deviations of 2 players, say players i and j. Here we denote $S^1 = (S_i, S_j, S_{-\{i,j\}})$, $S^2 = (S'_i, S_j, S_{-\{i,j\}})$, $S^3 = (S'_i, S'_j, S_{-\{i,j\}})$, and $S^4 = (S_i, S'_j, S_{-\{i,j\}})$. For the cycle $(S_1, S_2, S_3, S_4, S_1)$ consider the difference in personal cost of the moving players $\Delta_i^{12} = c_i(S^2) - c_i(S^1)$, $\Delta_j^{23} = c_j(S^3) - c_j(S^2)$, $\Delta_i^{34} = c_i(S^4) - c_i(S^3)$, $\Delta_j^{41} = c_j(S^1) - c_j(S^4)$. Note that existence of an exact potential function is equivalent to assuming that this difference is 0, i.e.,

$$\Delta_i^{12} + \Delta_j^{23} + \Delta_i^{34} + \Delta_j^{41} = 0 \quad , \tag{1}$$

for every pair of players i and j and every 4-tuple of states as detailed above [21].

Now suppose Γ is an exact potential game for every binary context F. Note that for 2 players, every exact potential game is isomorphic to a congestion game with affine delays, because each resource is used by at most 2 players. Hence, consider a game with at least three players. The main idea of the proof is to characterize the impact on the personal cost of player h when a different player i makes a strategy switch. Using this characterization, we then construct resources and affine delay functions.

Consider three different players $i, j, h \in K$ and F with $f_{ih} = f_{hi} = 1$ and 0 for all other pairs of players in the game. We assume that the resulting game has an exact potential, we have

$$\Delta_i^{12} + c_h(S^2) - c_h(S^1) + \Delta_j^{23} + \Delta_i^{34} + c_h(S^4) - c_h(S^3) + \Delta_j^{41} = 0 \ ,$$

and by using Eqn. (1) above and the definition of $S^1, \ldots, S^4$, we see that

$$c_h(S_i', S_j, S_{-\{i,j\}}) - c_h(S_i, S_j, S_{-\{i,j\}}) = c_h(S_i', S_j', S_{-\{i,j\}}) - c_h(S_i, S_j', S_{-\{i,j\}}) \ .$$

The sides of this equation describe the change of personal cost of h when i switches from S_i to S_i', once with j playing S_j (left) and once with j playing S_j' (right). We can derive this identity for all strategies of each player $j \neq i, h$. This shows that when i changes his strategy from S_i to S_i', then the change in personal cost of h is independent of the strategy of any other player j. Hence, there is

$$\Delta_h(S_i', S_i, S_h) = d_h(S_i', S_h, S_{-\{i,h\}}) - d_h(S_i, S_h, S_{-\{i,h\}}) \ .$$

To show that these values are pairwise consistent, we again consider F with $f_{ih} = f_{hi} = 1$ and 0 for all other pairs of players. However, this time i and h do the strategy switches. By considering a 4-cycle as above and using Eqn. (1), we obtain

$$\begin{aligned} &c_h(S_i', S_h, S_{-\{i,h\}}) - c_h(S_i, S_h, S_{-\{i,h\}}) \\ &+ c_i(S_i', S_h', S_{-\{i,h\}}) - c_i(S_i', S_h, S_{-\{i,h\}}) \\ &+ c_h(S_i, S_h', S_{-\{i,h\}}) - c_h(S_i', S_h', S_{-\{i,h\}}) \\ &+ c_i(S_i, S_h, S_{-\{i,h\}}) - c_i(S_i, S_h', S_{-\{i,h\}}) = 0 \ , \end{aligned}$$

or, equivalently,

$$\Delta_h(S_i', S_i, S_h) + \Delta_i(S_h', S_h, S_i') + \Delta_h(S_i, S_i', S_h') + \Delta_i(S_i, S_i', S_h) = 0 \ . \tag{2}$$

We now construct an equivalent congestion game Γ' with affine delay functions. We consider each pair of players $i \neq h$ and introduce a single resource r_{S_i,S_h} for every pair of strategies in $\mathcal{S}_h \times \mathcal{S}_i$. For strategy S_h, we assume that it contains all resources for which it appears in the index. Let us first restrict our attention to one pair of players i and h. Due to the fact that the values Δ_i and Δ_h can be given separately for each pair i, h and do not depend on other player strategies, we can effectively reduce the game to a set of 2-player games played simultaneously.

For each resource r associated with strategies of both i and h, we set all delays $d_r(1) = 0$. The delay $d_r(2)$ is set to 1 for one arbitrarily chosen resource

r_{S_h,S_i}. The other delays $d_r(2)$ simply are derived via the differences Δ_h and Δ_i. In particular, with $r' = r_{S'_i,S_h}$ and $r = r_{S_i,S_h}$ we have $d_{r'}(2) = d_r(2) + \Delta_h(S'_i, S_i, S_h)$. Similarly, with $r' = r_{S_i,S'_h}$ and $r = r_{S_i,S_h}$ we have $d_{r'}(2) = d_r(2) + \Delta_i(S'_h, S_h, S_i)$. The set of values $d_r(2)$ defined in this way is consistent, because Eqn (2) essentially proves existence of an exact potential function when differences are given by Δ_h and Δ_i values, as the sum of changes in all 4-cycles of the state graph is 0. By our assignment, we essentially use this potential function for the $d_r(2)$ values.

In our construction so far, we guarantee that in Γ' player i suffers from the same cost change as in Γ when *the other player moves*. So far, however, it does not necessarily implement the correct personal cost or cost change for *the moving player*. For this we introduce a single resource r_{S_i} for strategy $S_i \in \mathcal{S}_i$ of every player $i \in K$. This resource is used only by player i and only if he plays strategy S_i. We again set the delay $d_r(1) = 1$ for some arbitrary resource r_{S_i}. Then consider a state (S_i, S_{-i}) and the deviation to (S'_i, S_{-i}). The difference in cost for player i is denoted by $\Delta_i(S'_i, S_i, S_{-i})$, and with $r = r_{S_i}$, $r' = r_{S'_i}$, $R_{ij} = \{r_{S_i,S_j} \mid S_j \in \mathcal{S}_j\}$ and $R'_{ij} = \{r_{S'_i,S_j} \mid S_j \in \mathcal{S}_j\}$ we get

$$d_{r'}(1) = d_r(1) + \Delta_i(S'_i, S_i, S_{-i}) + \sum_{\substack{j \in K \\ j \neq i}} \left(\sum_{s \in R_{ij}} d_s(S_i, S_{-i}) - \sum_{s \in R'_{ij}} d_s(S'_i, S_{-i}) \right).$$

Thus, we simply account for all delay changes from the sets of resources R_{ij} and R'_{ij} and correct the cost to implement the correct delay change of $\Delta_i(S'_i, S_i, S_{-i})$ via our resource r_{S_i}. Note that this gives a consistent set of values for $d_r(1)$. For a fixed S_{-i}, this implies the same cost changes for i as in Γ. To show that this correctly implements all cost changes for player i as in Γ, consider the switch from S_i to S'_i for a different set of strategies S'_{-i} and the cost change $\Delta_i(S'_i, S_i, S'_{-i})$. To see that the correct cost change is present also in Γ', we implement the deviation via the following shift. We first let all players other then i change to S_{-i}. By construction this changes i's personal cost as in Γ. Then we let i deviate to S_i in state (S'_i, S_{-i}). This yields a change in personal cost as in Γ by definition. Afterwards, we let other players switch back to S'_{-i}. Again, the cost changes of player i are implemented as in Γ. Hence, in conclusion, by implementing the correct cost change $\Delta_i(S_i, S'_i, S_{-i})$ for a single strategy switch of S_i to S'_i, all other cost changes for switches among these strategies are uniquely and correctly determined.

This shows that we can turn Γ into a congestion game Γ' with the same potential function, in which every resource is accessed by at most two players. Trivially, for every such resource we can generate the required delays $d_r(1)$ and $d_r(2)$ via an affine delay function $d_r(x) = a_r \cdot x + b_r$. □

Lemma 2. *A congestion game with affine delay functions has a context-potential for every social context.*

Proof. The context-potential function is given by

$$\Phi(S,F) = \sum_{r\in R} \sum_{j=1}^{n_r(S)} d_r(j) + \sum_{\substack{i\neq j\in K,\\ r\in S_i\cap S_j}} f_{ij} a_r$$

In case of affine delays $d_r(x) = a_r \cdot x + b_r$, we can equivalently assume that all delays are linear $d_r(x) = a_r \cdot x$ by appropriate introduction of player-specific resources that account for the offsets b_r. Then, the change of cost for player j if i changes from S_i to S_i' is given by 0 for the resources of S_j that are used in neither or both S_i and S_i'. The change is a_r or $-a_r$ for each resource r that is joined or left by i, respectively. Hence, when we examine the potential, we see that

$$\begin{aligned}
&\Phi(S_i, S_{-i}, F) - \Phi(S_i', S_{-i}, F)\\
&= \Delta_i(S_i', S_i, S_{-i}) + \sum_{\substack{i\neq j\in K,\\ r\in S_i'\cap S_j}} f_{ij} a_r - \sum_{\substack{i\neq j\in K,\\ r\in S_i\cap S_j}} f_{ij} a_r\\
&= \Delta_i(S_i', S_i, S_{-i}) + \sum_{\substack{i\neq j\in K,\\ r\in (S_i'-S_i)\cap S_j}} f_{ij} a_r - \sum_{\substack{i\neq j\in K,\\ r\in (S_i-S_i')\cap S_j}} f_{ij} a_r\\
&= \Delta_i(S_i', S_i, S_{-i}) + \sum_{i\neq j\in K} f_{ij} \cdot \Delta_j(S_i', S_i, S_{-i}) \enspace ,
\end{aligned}$$

as desired. This proves the lemma. □

We note that even for binary social contexts, every context-potential game must be isomorphic to a congestion game with affine delays. In turn, our positive result is more general – congestion games with affine delays are context-potential games for arbitrary social contexts F.

3 Computational Results

In this section, we study the computational complexity of deciding existence of PNE in a given potential games with social context. Throughout this section, we focus on binary contexts. We will say that player i *is friends with* player j if $f_{ij} = f_{ji} = 1$.

3.1 Congestion Games

We first focus on congestion games as introduced above. For this central class of games we can prove a NP-completeness result even for singleton games, in which $|S_i| = 1$ for all players $i \in K$ and all strategies $S_i \in \mathcal{S}_i$. We start with a game that does not have a PNE. This game is then used below in our construction to show NP-completeness of the decision problem.

Example 1. Consider a congestion game Γ consisting of the set of players $K = \{1,2,3,4\}$ and the set of resources $R = \{r_1, r_2\}$. Players 1 and 2 have only one strategy each, with $\mathcal{S}_1 = \{\{r_1\}\}$ and $\mathcal{S}_2 = \{\{r_2\}\}$. Players 3 and 4 both have two strategies, $\mathcal{S}_3 = \mathcal{S}_4 = \{\{r_1\}, \{r_2\}\}$. Both resources have the same delay function d_r with $d_r(1) = 4$, $d_r(2) = 8$ and $d_r(3) = d_r(4) = 9$. The binary context is such that player 4 is friends with all other players. Every other player is only friends with player 4.

It is easy to verify that this game has no PNE: In a state in which both resources are used by two players, player 4 has an improvement move by moving to the other resources. In a strategy profile in which player 3 and 4 are both on the same resource, player 3 has an improvement move by moving to the other resource.

Theorem 2. *It is* NP*-complete to decide if a singleton congestion game with binary context has a pure Nash equilibrium.*

The previous result uses concave delay functions to construct a game without PNE. It is an open problem if PNE always exist in singleton congestion games with binary context and convex delays. For more general structures of strategy spaces, however, convex delay functions are not sufficient. Again, we use the example below to prove NP-completeness of deciding existence.

Example 2. We consider a congestion game with six players denoted by $K = \{1, \ldots, 6\}$. Player 1 is friends with 3 and 4. Player 2 is friends with 5 and 6. The set of resources is $R = \{r_1, r_2, r_3, r_4, r_5\}$. Players 1 and 2 have two strategies. The strategies of player 1 are $\mathcal{S}_1 = \{\{r_1\}, \{r_2, r_3\}\}$. The strategies of player 2 are $\mathcal{S}_2 = \{\{r_2, r_4\}, \{r_3, r_5\}\}$. The remaining players have one strategy each, $\mathcal{S}_3 = \{\{r_1\}\}$, $\mathcal{S}_4 = \{\{r_3\}\}$, $\mathcal{S}_5 = \{\{r_2\}\}$ and $\mathcal{S}_6 = \{\{r_5\}\}$.

Note that r_4 is used by at most 1 player, r_1, r_5 by at most 2 players each, r_2, r_3 by at most 3 players. We define the convex delays only for the required number of players. For r_1 we have $d_{r_1}(1) = 15$ and $d_{r_2}(2) = 16$. Resources r_2 and r_3 have the same delay function with $d_r(1) = 5.5$, $d_r(2) = 6$ and $d_r(3) = 10$. Resource r_4 has delay $d_{r_4}(1) = 1$. Finally, r_5 has delay $d_{r_5}(1) = 0$ and $d_{r_5}(2) = 1$.

Note that only players 1 and 2 have more than one strategy. Thus, to verify that this game does not have a PNE, we have to check the four possible states represented by the strategies of players 1 and 2. In state $(\{r_1\}, \{r_2, r_4\})$ the perceived cost of player 1 is $16 + 16 + 6 = 38$ and he would improve by changing to strategy $\{r_2, r_3\}$ resulting in perceived cost of $15 + 10 + 6 + 6 = 37$. In state $(\{r_2, r_3\}, \{r_2, r_4\})$, the perceived cost of player 2 is $10 + 10 + 1 + 0 = 21$ and he would improve by changing to strategy $\{r_3, r_5\}$ resulting in perceived cost of $10 + 1 + 6 + 1 = 18$. In state $(\{r_2, r_3\}, \{r_3, r_5\})$, the perceived cost of player 1 is $6 + 10 + 15 + 10 = 41$ and he would improve by changing to strategy $\{r_1\}$ resulting in perceived cost of $16 + 16 + 6 = 38$. In state $(\{r_1\}, \{r_3, r_5\})$, the perceived cost of player 2 is $6 + 1 + 5.5 + 1 = 13.5$ and he would improve by changing to strategy $\{r_2, r_4\}$ resulting in perceived cost of $6 + 1 + 6 + 0 = 13$.

Theorem 3. *It is* NP*-complete to decide if a general congestion game with binary context has a pure Nash equilibrium even if the delay functions are convex.*

As an extension to ordinary congestion games, we also consider weighted congestion games. In this case, each player $i \in K$ has a weight $w_i \in \mathbb{N}$. Instead of the *number* of players using resource r, the delay function d_r now takes the *sum of weights* of players using r as input and maps it to a delay value. The personal cost of a player is the sum of delays of chosen resources. Weighted congestion games are known to possess PNE for linear and exponential delay functions, see [15]. Here we show that with binary context, even singleton weighted congestion games with identical linear delays might not have PNE.

Example 3. Consider the following game on two identical resources. Each resource r has the delay function $d_r(x) = x$. The game consists of four players with weights 1,1,4, and 9, respectively. The binary context is such that the three players with weights 1 and 4 are all friends with each other, but the player with weight 9 is not friends with anyone. It is easy to verify that this game does not have a PNE.

Theorem 4. *It is* NP*-complete to decide if a weighted singleton congestion game with binary context has a pure Nash equilibrium even if all delay functions are linear.*

3.2 Cost Sharing

In this section, we consider several classes of cost sharing games. We first study *Shapley* or *fair cost-sharing games*. These games are congestion games with delay functions $d_r(x) = c_r/x$, where $c_r \in \mathbb{N}$ is the cost of the resource. In these games, the cost of a resource is assigned in equal shares to all players using the resource. As a subclass, we consider *broadcast games* with Shapley sharing in which there is a directed or undirected graph $G = (V, E)$ with a single sink node $t \in V$. Every edge $e \in E$ is a resource. Every node $v_i \in V$, $v_i \neq t$ is associated to a different player i. The strategy set $\mathcal{S}_i$ consists of all v_i-t-paths in G.

A different cost sharing scheme proposed in [6] yields *Prim cost-sharing games*. In this case, resources are edges of a directed or undirected graph $G = (V, E)$ and players are situated at a subset of the nodes in this graph. There is a single sink node t, and the set of strategies for a player i in node v_i is the set of v_i-t-paths in G. There is a global ranking of players and the cost of an edge is assigned fully to the highest ranked player using it. The ranking of players derives from the ordering, in which Prim's algorithm would add players to construct a minimum spanning tree (MST). In particular, the first player i is the one which has the cheapest path to t in G. The second player is the one, which has the cheapest path to $\{t, v_i\}$, and so on. Again, in a *broadcast game* with Prim sharing every node $v \neq t$ is associated with a different player.

We first show that Shapley cost-sharing games with binary context might not possess a PNE. Remarkably, this even holds for broadcast games with undirected edges as the following example shows. We then use this example game as a building block in our NP-completeness result for broadcast games with Shapley sharing and binary context.

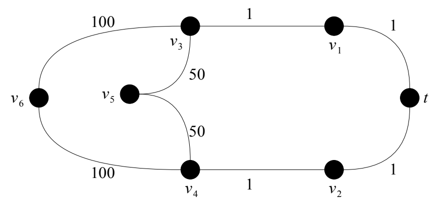

Fig. 1. A Shapley cost-sharing game that does not have a pure Nash equilibrium. The players v_3, v_4, and v_5 are friends.

Example 4. Consider a broadcast game with Shapley sharing in the network depicted in Figure 1. The edges are labeled with their costs. The players that belong to the vertices v_3, v_4, and v_5 are mutual friends. If player v_5 chooses the path via v_3 (or v_4), the best response of player v_6 is to choose his path via v_3 (or v_4, respectively). However, the best response of player v_5 is inverted. If player v_6 chooses the path via v_3 (or v_4), the best response of player v_5 is to choose his path via v_4 (or v_3, respectively). Thus, no PNE exists.

Theorem 5. *It is* NP*-complete to decide if a broadcast game with Shapley sharing and binary context has a pure Nash equilibrium.*

For Prim cost-sharing games existence and convergence results become more delicate. In particular, for undirected broadcast games with Prim sharing and arbitrary binary context there always exists a PNE. However, we first show that a similar result does not hold for directed broadcast games. The following example shows that such games with binary contexts do not have PNE in general. The main idea to prove existence of PNE and convergence without social context is that the player priorities induce a lexicographic potential function for the game. If we allow additive social context, the lexicographic improvement property breaks. This is then used to prove NP-completeness of deciding existence of PNE below.

Example 5. Figure 2 shows an example of a Prim cost-sharing game that does not have a PNE. In this game player d is friends with all other players and the players b and c are friends. Observe that in every state, d uses the edge of cost 1000. Hence, this cost is part of the perceived cost of every player in every state. Therefore, the players never have an incentive to use one of dashed edges. On the other hand, these are the edges that define the priorities of the players. Given their priorities, it is straightforward to verify that the players never agree on a subset of the edges of small cost to buy. Hence, no state of the game qualifies as a PNE. To turn this game into a broadcast game, note that we can safely add another player to every intermediate (non-filled) node. These players have only one strategy each, they will end up with lowest priority, and thus they do not change the cyclic incentives of players a, b, c and d described above.

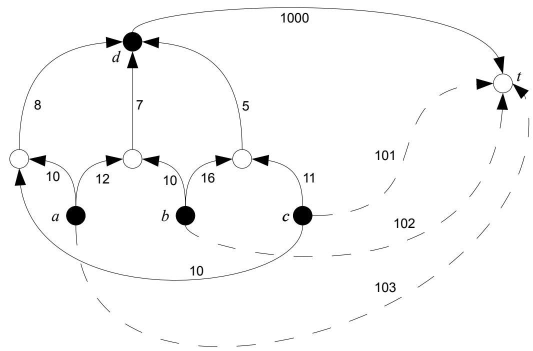

Fig. 2. An example of a Prim cost-sharing game with a binary context that does not have a pure Nash equilibrium. Here, player d is friends with a, b, and c and the players b and c are friends.

Theorem 6. *It is* NP*-complete to decide whether a directed broadcast game with Prim sharing has a pure Nash equilibrium.*

In contrast, if we consider undirected broadcast games with Prim sharing and binary contexts, we can construct a PNE using an efficient centralized assignment algorithm. While this shows existence of a PNE, convergence of improvement moves might still be absent. In fact, our theorem below shows the slightly stronger statement that these games are not even weakly acyclic.

Theorem 7. *For every undirected broadcast game with Prim cost sharing there is a pure Nash equilibrium if the social context F satisfies $f_{ij} = f_{ji} \in [0,1]$ for all $i, j \in K$. The pure Nash equilibrium can be computed in polynomial time.*

Proof. The proof of the theorem is mainly a consequence of classic arguments showing non-emptiness of the core in cooperative minimum spanning tree games. We here use Prim's MST algorithm not only to define the priority ordering of players but also to construct a PNE. We first consider the cheapest incident edge to t and assign the incident player v to play strategy $\{v, t\}$. Subsequently, consider the set V' of players connected to t. Consider the cheapest edge connecting a player of V' to a player in $V - V'$. We denote the players incident to this edge by $v' \in V'$ and $v \in V - V'$. Now we expand V' by assigning v to play the strategy composed of edge (v, v') and the path that v' uses to connect to t. This inductively constructs a state, in which the cost of a MST is shared. Note that the players are added in order of their priority, and hence every player pays exactly for the first edge on his path to t. We will argue that this state is a PNE for every social context with $f_{ij} = f_{ji} \in [0,1]$ for all $i, j \in K$.

Assume that a player i has a profitable strategy switch that decreases his perceived cost. This switch does not change the personal cost of any higher ranked player, these players will stay connected to t by sharing the cost of their subtree. In addition, the set of all players shares the cost of a MST, i.e., a minimum cost network connecting all players to t. Hence, the sum of all personal costs cannot decrease in a strategy switch. First, suppose the personal cost of player i strcitly decreases in the strategy switch. Note that all players connecting to t via his node v_i have lower priority. Hence, we could construct a cheaper network by letting all these players imitate i's strategy switch, because this would not change the personal cost of the imitating players. In this way, we would obtain a strictly cheaper network connecting all players to t, a contradiction.

Thus, the only way to improve the perceived cost is to strictly decrease the cost of other players that he is friends with. However, player i can only decrease the cost of lower ranked players by paying some of the edges currently assigned to them. As $f_{ij} = f_{ji} \leq 1$, he obtains no benefit from paying these edges himself. As $f_{ij} = f_{ji} \geq 0$, he obtains no benefit from forcing lower ranked players to pay the edges he vacates. Hence, if he strictly lowers his perceived cost in this way, then he must also strcitly decrease his personal cost, which is impossible as noted above. □

Theorem 8. *There is an undirected broadcast games with Prim sharing and binary context with the property that there exists a starting state from which there is no sequence of improvement moves to a PNE.*

Proof. We construct an example game and an appropriate starting state. Our game is an adaptation of the game in Fig. 2. We simply turn every directed edge into an undirected edge. The social context is as before, but here we also assume that the three auxiliary players in non-filled nodes are all friends with d. In our starting state, player d uses the edge of cost 1000, and all other players use some cycle-free path to t that goes over node d. The main invariant is that players c and b always remain on the edge of cost 1000. Given this condition, player c has no incentive to switch to a path containing an edge of cost 101, because otherwise b would be assigned to pay a cost of 1000. If c is assigned to pay the cost of 1000, all players have an incentive to join c on this edge as the corresponding paths become cheaper. Thus, no player will have an improvement move purchasing some of the edges of cost 101, 102 or 103. However, it is straightforward to verify that without these edges, no PNE can be obtained, and hence no sequence of improvement moves leads to a PNE. □

3.3 Market Sharing Games

Market sharing games are a class of congestion games that model content distribution in ad-hoc networks. There is a set of players and a set of markets. Each player i has a budget B_i, each market has a cost C_i. In addition, a market has a query rate q_i. There is a bipartite network specifying which player can participate in which market. From the set of markets a player is connected to, he can choose as strategy any subset for which the sum of costs is at most his budget. The reward from a market is the query rate, and it is shared equally by the players that pick the market. Every player gets as utility the sum of rewards of markets chosen in his strategy. More generally, market games are congestion games with utility-maximizing players and reward functions $d_r(x) = q_r/x$. Market costs and budgets determine the structure of the strategy spaces.

In market sharing games with binary context, we again observe absence of PNE and NP-completeness of deciding PNE existence.

Example 6. Consider the following market sharing game with two identical markets. Each market has cost of 1 and its query rate (revenue) is 1. There are four players 1, 2, 3, and 4 in this game. Each player is interested in both markets and each player has a budget of 1. The players 1, 2, and 3 are mutual friends. It is easy to see that this game does not have an equilibrium. The players 1, 2, and 3 prefer an outcome in which one of them is in a market by himself.

Theorem 9. *It is* NP*-complete to decide if a market sharing game with a binary context has a pure Nash equilibrium.*

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A Missing Proofs

A.1 Proof of Theorem 2

Proof. We reduce from 3SAT. Without loss of generality [24], we assume each variable appears at most twice positively and at most twice negatively. Given a formula φ, we construct a congestion game G_φ that has a PNE if and only if φ is satisfiable. Let $x_1, \ldots, x_n$ denote the variables and $c_1, \ldots, c_m$ the clauses of a formula φ.

For each variable x_i there is a player X_i that chooses one of the resources $r^1_{x_i}$, $r^0_{x_i}$, or r_0. The resources $r^1_{x_i}$ and $r^0_{x_i}$ have the delay function $9x$ and resource r_0 has the delay function $7x+3$.

For each clause c_j, there is a player C_j who can choose one of the following resources. For every positive literal x_i in c_j he may choose $r^0_{x_i}$. For every negated literal $\neg x_i$ in c_j he may choose $r^1_{x_i}$. Note that there is a stable configuration with no variable player on r_0 if and only if there is a satisfiable assignment for φ.

Additionally, there are four players u_0, u_1, u_2, and u_3 and two resource r_1 and r_2. Each of the resources r_1 and r_2 has delay 4 if used by one player, delay 8 if used by two players and delay 9 otherwise. Player u_1 is always on resource r_1 and player u_2 is on resource r_2. The players u_4 choose between r_1 and r_2. Player u_0 chooses between r_1, r_2, and r_0. Player u_0 is friend with each of the players u_1, u_2, and u_3. These are the only friendship relations in this game. Note that player u_0 choses r_1 or r_2 if one of the variable agents is on r_0.

If φ is satisfiable by an assignment $(x^*_1, \ldots, x^*_n)$, a stable solution for G_φ can be obtained by placing each variable player X_i on $r^{x^*_i}_{x_i}$. Since $(x^*_1, \ldots, x^*_n)$ satisfies φ there is one resource for each clause player that is not used by a variable player. Thus, we can place each clause player on this resource, which he then shares with at most one other clause player. Let u_0 use r_0 and u_1 and u_2 choose r_1 and u_3 choose r_2. It is easy to check that this is a PNE.

If φ is unsatisfiable, there is no stable solution. To prove this it suffices to show that one of the variable players prefers r_0. In that case player u_0 never chooses r_0 and the players $u_0, \ldots, u_3$ essentially play the game of Example 1. For the purpose of contradiction assume that φ is not satisfiable but there is a stable solution in which no variable player wants to choose r_0. This implies that there is no other player, i.e. a clause player, on a resource that is used by a variable player. However, if all clause players are on a resource without a variable player we can derive a corresponding bit assignment which, by construction, satisfies φ.

Therefore, G_φ has a stable solution if and only if φ is satisfiable. □

A.2 Proof of Theorem 3

Proof. We reduce from 3SAT. Given a formula φ, we construct a congestion game G_φ that has a PNE if and only if φ is satisfiable. Let $x_1, \ldots, x_n$ denote the variables and $c_1, \ldots, c_m$ the clauses of a formula φ. Without loss of generality [24], we assume each variable appears at most twice positively and at most twice negatively.

For each variable x_i there is a player X_i that chooses one of the resources $r^1_{x_i}$, $r^0_{x_i}$, or r_0. The resources $r^1_{x_i}$ and $r^0_{x_i}$ have the delay function $9x$ and resource r_0 has the delay function $7x + 3$.

For each clause c_j, there is a player C_j who can choose one of the following three resources. For every positive literal x_i in c_j he may choose $r^0_{x_i}$. For every negated literal $\bar{x}_i$ in c_j he may choose $r^1_{x_i}$. Note that there is a stable configuration with no variable player on r_0 if and only if there is a satisfiable assignment for φ.

Additionally, there are six players $u_1, \ldots, u_6$ and five resources r_1, r_2, r_3, r_4, r_5.

The delay functions are as follows.

Resource r_1 delay of 15 for one player and delay of 16 for two or more players. Resources r_2 and r_3 delay of 5.5 for one player, delay of 6 for two players and delay of 10 for three or more players. Resource r_4 has delay of 1. Resource r_5 has delay of 0 for one player and delay of 1 for two or more players.

The strategies of player u_1 are $\mathcal{S}_1 = \{\{r_1\}, \{r_2, r_3\}\}$. The strategies of player u_2 are $\mathcal{S}_2 = \{\{r_2, r_4\}, \{r_3, r_5\}\}$. The strategies of player u_3 are $\mathcal{S}_3 = \{\{r_1\}, \{r_0\}\}$, The remaining players have one strategy each, $\mathcal{S}_4 = \{\{r_3\}\}$, $\mathcal{S}_5 = \{\{r_2\}\}$ and $\mathcal{S}_6 = \{\{r_5\}\}$.

Player u_1 is friend with u_3 and u_4. Player u_2 is friend with u_5 and u_6. These are the only friendship relations in this game. Note, that the players $u_1, \ldots, u_6$ essentially play the game described in Example 2 if player u_3 never chooses strategy $\{r_0\}$.

If φ is satisfiable by an assignment $(x^*_1, \ldots, x^*_n)$, a stable solution for G_φ can be obtained by placing each variable player x_i on $r^{x^*_i}_{x_i}$. Since $(x^*_1, \ldots, x^*_n)$ satisfies φ there is one resource for each clause player that is not used by a variable player. Thus, we can place each clause player on this resource, which he then shares with at most one other clause player. Let u_1 play $\{r_1\}$, u_2 play $\{r_2, r_4\}$, and u_3 play $\{r_0\}$. It is easy to check that this is a PNE.

If φ is unsatisfiable, there is no stable solution. To prove this it suffices to show that one of the variable players prefers r_0. In that case player u_3 never chooses $\{r_0\}$ and the players $u_0, \ldots, u_6$ play the sub game of Example 2. For the purpose of contradiction assume that φ is not satisfiable but there is a stable solution in which no variable player wants to choose r_0. This implies that there is no other player, i.e., a clause player, on a resource that is used by a variable player. However, if all clause players are on a resource without a variable player we can derive a corresponding bit assignment which, by construction, satisfies φ.

Therefore, G_φ has a stable solution if and only if φ is satisfiable. □

A.3 Proof of Theorem 4

Proof. We reduce from 3SAT. Without loss of generality [24], we assume each variable appears at most twice positively and at most twice negatively. Given a formula φ, we construct a congestion game G_φ that has a PNE if and only if φ is satisfiable. Let $x_1, \ldots, x_n$ denote the variables and $c_1, \ldots, c_m$ the clauses of φ.

For each variable x_i $(1 \leq i \leq n)$ there are two resources Y_i and $\bar{Y}_i$. For each clause c_j $(1 \leq i \leq n)$ there are three resources Z_j, K_j^1, and K_j^2. Each resource has the delay function $d(x) = x$.

For each variable x_i there is a player $\tilde{x}_i$ with weight 1000 that chooses either Y_i or $\bar{Y}_i$. For each clause j that contains the positive literal x_i, there is a player $x_{i,j}$ with weight 100. His first strategy is $\{Y_i\}$ and his second strategy is $\{Z_j\}$. For each clause j that contains the negated literal $\neg x_i$, there is a player $\bar{x}_{i,j}$ with weight 100. His first strategy is $\{\bar{Y}_i\}$ and his second strategy is $\{Z_j\}$.

For each clause c_j, there are eight players. There is player $\tilde{c_j}$ with weight 20. He chooses either Z_j or K_j^1. There are four players $k_j^1, \dots, k_j^4$ with weights 1, 1, 4, and 9, respectively. They choose between resource K_j^1 and K_j^2. The binary context consists of m cliques of friends among the players k_j^1, k_j^2, and k_j^3, respectively. Finally, there are three dummy players of weight 500, 500, and 499 that are on resource K_j^1, K_j^2, and Z_j respectively.

If ϕ is satisfiable with $\chi_1, \dots, \chi_n$, the following strategy is a PNE: For $1 \leq i \leq n$, let player $\tilde{x}_i$ allocate resource Y_i if χ_i is true, and $\bar{Y}_i$ otherwise. For $1 \leq i \leq n$ if χ_i is true, the players $x_{i,j}$ play their second strategy and the players $\bar{x}_{i,j}$ play their first strategy. If χ_i is false, the players $\bar{x}_{i,j}$ play their second strategy and the players $x_{i,j}$ play their first strategy. For $1 \leq j \leq m$, each player $\tilde{c}_j$ plays strategy K_j^1 and players $k_j^1, \dots, k_j^4$ play K_j^2. One can easily check that this is indeed a PNE.

For the sake of contradiction assume ϕ is not satisfiable but there is a PNE. Let $\chi_1, \dots, \chi_n$ be a truth assignment defined as follows: $\chi_i = 1$ if and only if player $\tilde{x}_i$ plays $\{Y_i\}$. In a PNE the players $x_{i,j}$ (and $\bar{x}_{i,j}$) play their first strategy if and only if $\chi_i = 1$ ($\chi_i = 0$, respectively). Now pick a clause c_j that is not satisfied by χ. The best response of $\tilde{c}_j$ is $\{Z_j\}$ since there is none of the variable players on Z_j. The players $k_j^1, \dots, k_j^4$ play the game of Example 3 on the resources K_j^1 and K_j^2 which does not have an equilibrium which yields the contradiction. □

A.4 Proof of Theorem 5

Proof. We reduce from 3SAT. Without loss of generality [24], we assume each variable appears at most twice positively and at most twice negatively. Given a formula ϕ with variables $X_1, \dots, X_n$ and clauses $C_1, \dots, C_m$, we construct a broadcast game G_ϕ that has an equilibrium if and only if the ϕ is satisfiable.

Before we describe the reduction in detail, let us briefly outline the basic ideas of the construction. See Figure 3 for reference. The game is consists of n gadgets for the variables and m gadgets for the clauses. For each clause there is a gadget similar to the game of Example 4. Note that player c_j was denoted by v_6 there. In G_ϕ vertex c_j of each clause gadget has three additional outgoing edges that connect to a variable gadget each. This gives player c_j three more paths to choose from compared to Figure 3. If he does not use one of those, the players of the clause gadget essentially play the game of Example 4 which does not have an equilbrium. However, if the players in a connected variable gadget

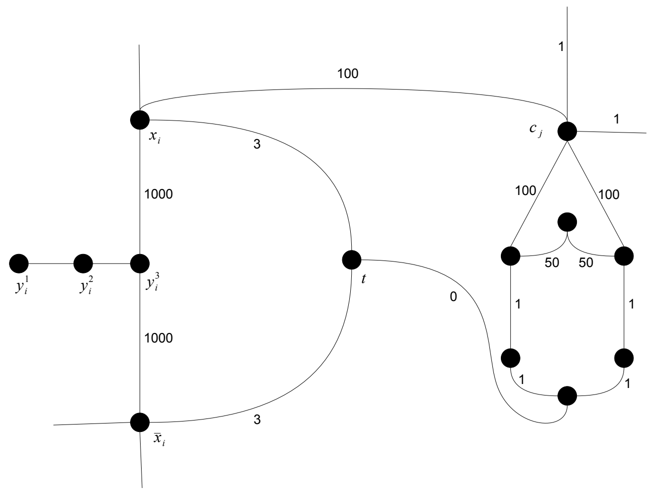

Fig. 3. Two gadgets of a Shapley cost-sharing game G_ϕ. On the left: A variable gadget of an variable x_i. On the right: A clause gadget that contains the literal x_i.

play a profile which corresponds to a satisfying assignment, player c_j may leave the clause gadget using the additional edge. The remaining players of the clause gadget then have equilibrium strategies.

Let us now describe the reduction in more detail. For each variable X_i with $i \in \{1, \dots, n\}$, there is *variable gadget* that consists of players $y_i^1, y_i^2, y_i^3, x_i, \bar{x}_i$ The subgraph which connect theses players and the target node t is depicted on the left hand side of Figure 3.

For each clause C_j with $j \in \{1, \dots, m\}$ there is a *clause gadget* depicted on the right hand side of Figure 3. For each literal of C_j there is a connection from the vertex c_j to a variable gadget. If X_i is a literal in C_j, vertex c_j is connected to x_i with an edge of weight 100. If $\neg X_i$ is a literal in C_j, vertex c_j is connected to $\bar{x}_i$ with an edge of weight 100 . The only friendship relations in this games are the three pair of friends in each clause gadget as in Example 4.

It remains to show that G_ϕ has a PNE if and only of ϕ is satisfiable.

Note that in a PNE the players y_i^1, y_i^2, y_i^3 use the same path from vertex y_i^3 onwards. Choosing the upper path via vertex x_i corresponds to X_i is set to true. Likewise, choosing the lower via vertex $\bar{x}_i$ corresponds X_i is set to false.

Observe that for a player c_j the cost of choosing a path via a vertex x_i (or $\bar{x}_i$) is higher than the cost of any path in the clause gadget if none of the players y_i^1, y_i^2, y_i^3 choose the path via vertex x_i (or $\bar{x}_i$, respectively). Note that at most one other clause player and the player x_i (or $\bar{x}_i$) can use this edge.

On the other hand, the cost of a path a via vertex x_i (or $\bar{x}_i$) is lower that the cost of any path in the clause gadget if all three players y_i^1, y_i^2, y_i^3 choose the path via vertex x_i (or $\bar{x}_i$, respectively).

Therefore, if the formula is satisfiable, there is an equilibrium by letting the players of the variable gadget play according to a satisfying assignment and the clause players choosing a path via a vertex of a variable that satisfy this clause. Conversely, if there is no satisfying assignment, there is a clause player whose best response is a path in the clause gadget. The players of the clause gadget play the game of Example 4 that does not have an equilibrium. □

A.5 Proof of Theorem 6

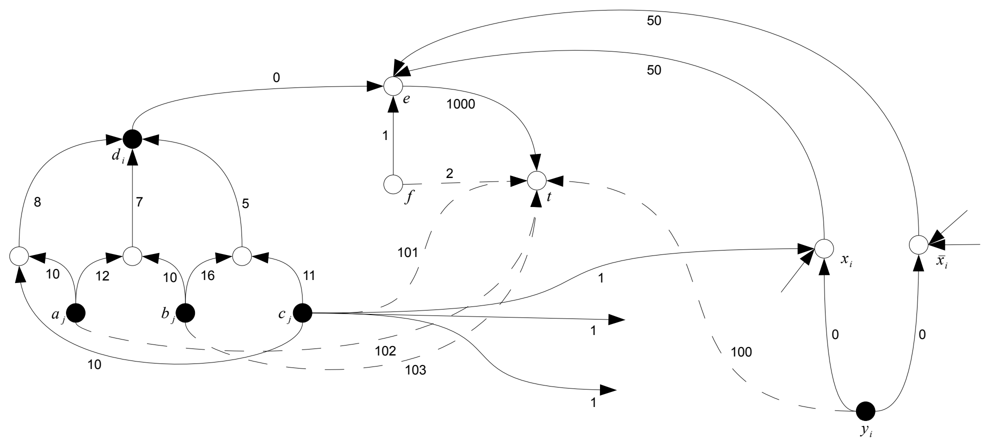

Fig. 4. A clause gadget (left) and a variable gadget (right) of a Prim cost-sharing game G_ϕ.

Proof. We reduce from 3SAT. Given a formula ϕ with variables $X_1, \ldots, X_n$ and clauses $C_1, \ldots, C_m$, we construct a cost sharing game G_ϕ that has an equilibrium if and only if the ϕ is satisfiable. See Figure 4 for reference. Again, we will make sure that the players do not have an incentive to use the dashed edges.

The game consists of three nodes t, e, f and of one gadget for every clause and one gadget for every variable in ϕ. Player f and player e are friends. Observe that it is always a best response for f to choose the path via e and pay for the edge of cost 1000.

For every clause C_j there is a *clause gadget* which is essentially a copy of the example given in Figure 2. However, the edge from d to t of cost 1000 in the original network is now replaced by an edge of cost 0 to e. Here, only the players b_j and c_j are friends.

For every variable X_i there is a *variable gadget* which consists of three players y_i, x_i and $\bar{x}_i$. Player y_i has essentially (ignoring the dashed edges) two paths he can choose from. One via a vertex x_i and one via $\bar{x}_i$. The clause and variable gadgets are connected by edges in the following way. If clause C_j contains the literal X_i ($\neg X_i$) there is a edge from vertex c_j to x_i ($\bar{x}_i$, respectively).

Now, if there is a satisfying assignment $\chi_1, \ldots, \chi_n$ for ϕ, one can obtain an equilibrium as follows. Player f chooses the path via e and pays for the edge of cost 1000. For every positive (negative) variable χ_i, let player y_i choose the path via x_i ($\bar{x}_i$, respectively). For every clause C_j, let player c_j choose the path via x_{i_j} (x_{i_j}) where i_j is the index of a positive (negated) literal X_{i_j} ($\neg X_{i_j}$, respectively) that satisfies clause C_j in χ. Note, that the edge of cost 50 is being payed by y_{i_j}. The players a_j choose the paths starting with the edges of cost 12 and the players b_j the paths starting with the edges of cost 10.

For the sake of contradiction, assume that ϕ is unsatisfiable and there is a PNE. As argued above, f chooses the path via e and pays for there edge of cost 1000. Obviously, none of the dashed edges is used in a PNE as there is always a cheaper path. We define a bit assignment χ as follows. Let $\chi_i = 1$ if player y_i choose the path via x_i and $\chi_i = 0$, otherwise. Let j be the index of a clause that is not satisfied by χ. Then player c_j would have to pay for every edge of cost 50 if he chooses a path via a variable gadget. Thus, the players of this clause gadget play the game of Example 5 which does not have an equilibrium.

□

A.6 Proof of Theorem 9

Proof. We present a similar reduction from 3SAT as in Theorem 2. Without loss of generality [24], we assume each variable appears at most twice positively and at most twice negatively. Given a formula φ, we construct a market sharing game G_φ that has a PNE if and only if φ is satisfiable. Let $x_1, \ldots, x_n$ denote the variables and $c_1, \ldots, c_m$ the clauses of a formula φ.

For each variable x_i there a player X_i with budget 10 that chooses one of the markets $r^1_{x_i}$, $r^0_{x_i}$, or $\{r_0, r^a_i\}$. The markets $r^1_{x_i}$ and $r^0_{x_i}$ each have query rate and cost 10. Markets r_0 and r^a_i have query rate 4 and cost 5.

For each clause c_j, there is a player C_j with budget 10 who can choose one of the following three markets. For every positive literal x_i in c_j he may choose $r^0_{x_i}$. For every negated literal $\neg x_i$ in c_j he may choose $r^1_{x_i}$. Note that there is a stable configuration with no variable player on r_0 if and only if there is a satisfiable assignment for φ.

Additionally, there are four players u_0, u_1, u_2, and u_3 and two markets r_1 and r_2. Each of the markets r_1 and r_2 has query rate and cost 5. Each player is interested in both markets and each player has a budget of 5. The players u_1, u_2, and u_3 are mutual friends. These are the only friendship relations in this game.

Note that player u_0 choses r_1 or r_2 if one of the variable agents is on r_0, in which case there is no PNE in the game.

If φ is satisfiable by an assignment $(x^*_1, \ldots, x^*_n)$, a stable solution for G_φ can be obtained by placing each variable player X_i on $r^{x^*_i}_{x_i}$. Since $(x^*_1, \ldots, x^*_n)$ satisfies φ there is one market for each clause player that is not used by a variable player.

Thus, we can place each clause player on this market, which he then shares with at most one other clause player. Let u_0 use r_0 and u_1 and u_2 choose r_1 and u_3 choose r_2. It is easy to check that this is a PNE.

If φ is unsatisfiable, there is no stable solution. To prove this it suffices to show that one of the variable players prefers $\{r_0, r_i^a\}$. In that case player u_0 never chooses r_0 and the players $u_0, \ldots, u_3$ essentially play the game of Example 6. For the purpose of contradiction assume that φ is not satisfiable but there is a stable solution in which no variable player wants to choose r_0. This implies that there is no other player, i.e. a clause player, on a market that is used by a variable player. However, if all clause players are on a market without a variable player we can derive a corresponding bit assignment which, by construction, satisfies φ.

Therefore, G_φ has a stable solution if and only if φ is satisfiable. $\square$