Non-cooperative Facility Location and Covering Games*

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Abstract. We study a general class of non-cooperative games coming from combinatorial covering and facility location problems. A game for k players is based on an integer programming formulation. Each player wants to satisfy a subset of the constraints. Variables represent resources, which are available in costly integer units and must be bought. The cost can be shared arbitrarily between players. Once a unit is bought, it can be used by all players to satisfy their constraints. In general the cost of pure-strategy Nash equilibria in this game can be prohibitively high, as both prices of anarchy and stability are in $\Theta(k)$. In addition, deciding the existence of pure Nash equilibria is NP-hard. These results extend to recently studied single-source connection games. Under certain conditions, however, cheap Nash equilibria exist: if the integrality gap of the underlying integer program is 1 and in the case of single constraint players. In addition, we present algorithms that compute cheap approximate Nash equilibria in polynomial time.

1 Introduction

Analyzing computational environments using game-theoretic models is a quickly evolving research direction in theoretical computer science. Motivated in large parts by the Internet, the resulting dynamics of introducing selfish behavior of distributed agents into a computational environment are studied. In this paper we follow this line of research by considering a general class of non-cooperative games based on general integer covering problems. Problems concerning service installation or clustering, which play an important role in large networks like the Internet, are modeled formally as some variant of covering or partition problems. Our games can serve as a basis to analyze these problems in the presence of independent non-cooperative selfish agents.

The formulation of our games generalizes an approach by Anshelevich et al [2], who proposed games in the setting of Steiner forest design. In particular, we consider a covering optimization problem given as an integer linear program and turn this into a non-cooperative game as follows. Each of the k non-cooperative players considers a subset of the constraints and strives to satisfy them. Each variable represents a resource, and integer units of resources can be bought by the players. The cost of a unit is given by the coefficient in the objective function. In particular, players pick as strategy a payment function that specifies how much they are willing to pay for the units

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of each resource. A unit is considered *bought* if the cost is paid for by the amount the players offer. Bought units can then be used by *all* players simultaneously to satisfy their constraints – no matter whether they contribute to the cost or not. A player strives to minimize the sum of her offers, but insists on satisfaction of her constraints. A variety of integer covering problems, most prominently variants of set cover and facility location, can be turned into a game with the help of this model. We study our games with respect to the existence and cost of stable outcomes of the game, which are exact and approximate Nash equilibria. At first, we characterize prices of anarchy [14] and stability [1]. They measure the social cost of the worst and best Nash equilibria in terms of the cost of a social optimum solution. Note that a social optimum solution is the optimum solution to the underlying integer program. As the cost of exact Nash equilibria can be as high as $\Theta(k)$, we then consider a two-parameter optimization problem to find (α, β)-approximate Nash equilibria. These are solutions in which each player can reduce her contribution by at most a factor of α by unilaterally switching to another strategy, and which represent a β-approximation to the socially optimum cost. We refer to α as the *stability ratio* and β as the *approximation ratio*.

Related Work *Competitive location* is an active research area, in which game-theoretic models for spatial and graph-based facility location have been studied in the last decades [7, 18]. These models consider facility owners as players that selfishly decide where to place and open a facility. Clients are modeled as part of player utility, e.g. they are always assumed to connect to the closest facility. Recent examples of this kind of location games are also found in [5, 21]. According to our knowledge, however, none of these models consider the clients as players that need to create connections and facilities without central coordination.

Closer to our approach are cooperative games and mechanism design problems based on optimization. In [6] strategyproof cost sharing mechanisms have been presented for games based on set cover and facility location. For set cover games this work was extended in [15,20] by considering different social desiderata and games with items or sets being agents. Furthermore, in [11] lower bounds on budget-balance for cross-monotonic cost sharing schemes were investigated. Cooperative games based on integer covering/packing problems were studied in [4]. It was shown that the core of such games is non-empty if and only if the integrality gap is 1. In [8] similar results are shown for a class of facility location games and an appropriate integer programming formulation. Cooperative games and the mechanism design framework are used to model selfish service receivers who can either cooperate to an offered cost sharing or manipulate. Our game, however, is strategic and non-cooperative in nature and allows players a much richer set of actions. We investigate distributed uncoordinated covering scenarios rather than a coordinated environment with a mechanism choosing customers, providing service and charging costs. Our model is suited for a case in which players have to directly invest into specific resources. Nevertheless our model has some connections to the cooperative setting, which we will outline in the end of Sect. 2.1.

The non-cooperative model we consider stems from [2], who proposed a game based on the Steiner forest problem. They show that prices of anarchy and stability are in $\Theta(k)$ and give a polynomial time algorithm for $(4.65 + \epsilon, 2)$-approximate Nash equilibria. In our uncapacitated facility location (UFL) game we assume that each of the clients must be connected directly to a facility. We can introduce a source node s, connect all facilities f to it, and direct all edges from clients to facilities. The costs for the new edges (f, s) are given by the opening costs $c(f)$ of the corresponding facilities. This creates a single-source connection game on a directed graph. If we allow indirect connections to facilities, the game can be turned into an undirected single-source connection game (SSC) considered in [2, 10]. For both UFL and SSC games results in [2] suggest that the price of anarchy is k and the price of stability is 1 if each player has a single client. Algorithms for $(3.1 + \epsilon, 1.55)$-approximate Nash equilibria in the SSC game were proposed in [10]. In a very recent paper [3] we considered our game model for the special case of vertex covering. Prices of anarchy and stability are in $\Theta(k)$ and there is an efficient algorithm computing $(2, 2)$-approximate Nash equilibria. For a lower bound it was shown that both factors are essentially tight. In addition, for games on bipartite graphs and games with single edge players the price of stability was shown to be 1. This paper extends and adjusts these results to a much larger class of games based on general covering and facility location problems.

Our results We study our games with respect to the quality and existence of pure strategy exact and approximate Nash equilibria. We will not consider mixed equilibria, as our model requires concrete investments rather than a randomized action, which would be the result of a mixed strategy. Our contributions are as follows.
Section 2 introduces the facility location games. Even for the most simple variant, the metric UFL game, the price of anarchy is exactly k and the price of stability is at least $k - 2$. Furthermore, it is NP-hard to determine whether a game has a Nash equilibrium. For the metric UFL game there is an algorithm to compute $(3, 3)$-approximate Nash equilibria in polynomial time. There is a lower bound of 1.097 on the stability ratio. For the more general class of facility location problems considered in [8] the price of stability is 1 if the integrality gap of a special LP-relaxation is 1. The best Nash equilibrium can be derived from the optimum solution to the LP-dual. Furthermore, if every player has only a single client, the price of stability is 1. We translate the lower bounds from the UFL game to SSC games [2] showing that it is NP-hard to determine Nash equilibrium existence and the price of stability is at least $k - 2$. In addition, there is a lower bound of 1.0719 for the stability ratio in the SSC game. This negatively resolves the question we left open in [10] whether the price of stability is 1 for SSC games with more than two terminals per player.
In Section 3 we consider general covering games. Even for the case of vertex cover it has been shown in [3] that prices of anarchy and stability are k and at least $k-1$, respectively, and it is NP-hard to decide the existence of exact Nash equilibria. We show that for set cover games, in which the integrality gap of the CIP-formulation is 1, the price of stability is 1. The best Nash equilibrium can be derived from the optimum solution to the LP-dual in polynomial time. If each player holds one item, the price of stability in set multi-cover games is 1. There is an algorithm to get $(\mathcal{F}, \mathcal{F})$-approximate Nash equilibria in set cover games in polynomial time, where $\mathcal{F}$ is the maximum frequency of any item in the sets. This generalizes results for vertex cover games on bipartite graphs and an algorithm for $(2, 2)$-approximate Nash equilibria for general vertex cover games [3]. Proofs omitted from this extended abstract will be given in the full version of the paper.

2 Facility Location Games

Consider the following non-cooperative game for the basic problem of uncapacitated facility location (UFL). Throughout the paper we denote a feasible solution by $\mathcal{S}$ and the social optimum solution by $\mathcal{S}^*$.
A complete bipartite graph $G = (T \cup F, T \times F)$ with vertex sets F of n_f *facilities* and T of n_t *clients* or *terminals* is given. Each of the k non-cooperative players holds a set $T_i \subset T$ of terminals. Each facility $f \in F$ has nonnegative opening costs $c(f)$, and for each terminal t and each facility f there is a nonnegative connection cost $c(t, f)$. The goal of each player is to connect her terminals to opened facilities at the minimum cost. Consider an integer programming (IP) formulation of the UFL problem:

$$\begin{array}{lll} \text{Min} & \sum_{f \in F} c(f) y_f + \sum_{t \in T} c(t,f) x_{tf} & \\ \text{subject to} & \sum_{f \in F} x_{tf} \geq 1 & \text{for all } t \in T \\ & y_f - x_{tf} \geq 0 & \text{for all } t \in T, f \in F \\ & y_f, x_{tf} \in \{0,1\} & \text{for all } t \in T, f \in F. \end{array} \tag{1}$$

Each player insists on satisfying the constraints corresponding to her terminals $t \in T_i$. She offers money to the connection and opening costs by picking as a *strategy* a pair of two *payment functions* $p_i^c : T \times F \rightarrow \mathbb{R}_0^+$ and $p_i^o : F \rightarrow \mathbb{R}_0^+$, which specify her contributions to the connection and opening costs, resp. These are her offers to the cost of raising the x_{tf} and y_f variables. If the total offer of all players exceeds the cost coefficient in the objective function (e.g. for a facility $\sum_i p_i(f) \geq c(f)$), the variable is raised to 1. In this case the corresponding connection or facility is considered bought or opened, resp. This affects *all* constraints, as all players can use bought connections and opened facilities for free, no matter whether they contribute to the cost or not. A *payment scheme* is a vector of strategies specifying for each player a single strategy. An (α, β)-*approximate Nash equilibrium* is a payment scheme in which no player can reduce her payments by more than a factor of α by unilaterally switching to another strategy, and which purchases a β-approximation to the socially optimum solution $\mathcal{S}^*$. We refer to α as the *stability ratio* and β as the *approximation ratio*. Using this concept a payment scheme purchasing $\mathcal{S}^*$ is an $(\alpha, 1)$-approximate Nash equilibrium, and an exact Nash equilibrium is $(1, \beta)$-approximate.
The following observations can be used to simplify a game. Suppose a terminal is not included in any of the terminal sets T_i. This terminal is not considered by any player and has no influence on the game. Hence, we will assume that $T = \bigcup_{i=1}^k T_i$.
Suppose a terminal t is owned by a player i and a set of players J, i.e. $t \in T_i \cap (\bigcap_{j \in J} T_j)$. Now consider an (approximate) Nash equilibrium for an adjusted game in which t is owned only by i. If t is added to T_j again, the covering requirement of player j increases. Contributions of j to resource units satisfying the constraint of t might have been superfluous previously, but become mandatory now as t is included in T_j. Thus j's incentive to deviate to another strategy does not increase. So if the payment scheme is an (α, β)-approximate Nash equilibrium for the adjusted game, it can yield only a smaller stability ratio for the original game. We will thus assume that all terminal

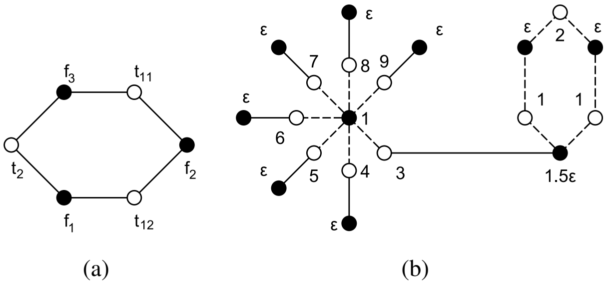

Fig. 1. (a) A metric UFL game without Nash equilibria – player 1 owns terminals labeled t_{11} and t_{12}, player 2 owns terminal t_2; (b) a metric UFL game with price of stability close to $k - 2$ for small ϵ – terminal labels indicate player ownership, facility labels specify opening costs. Black vertices are facilities, white vertices are terminals. All solid edges have cost 1, all dashed edges cost $\epsilon > 0$, all other edge costs are given by the shortest path metric.

sets T_i are mutually disjoint, as our results continue to hold if the sets T_i are allowed to overlap. Note that in any Nash equilibrium for such a game players do not share connection costs.

2.1 Metric UFL games

In this section we present results on exact and approximate Nash equilibria for the metric UFL game. For lower bound constructions we only consider a subset of *basic* edges, for which we explicitly specify the connection cost. All other edge costs are given by the shortest path metric over basic edges.
Even in the metric UFL game the price of anarchy is exactly k. The lower bound is derived by an instance with two facilities, f_1 with $c(f_1) = k$ and f_2 with $c(f_2) = 1$. Each player i has one terminal t_i, and all connection costs are $\epsilon > 0$. If each player pays a cost of 1 for f_1 and her connection cost, then no player has an incentive to switch and purchase f_2 completely. $\mathcal{S}^*$ is derived by opening only f_2 and connecting all terminals to it. This yields a lower bound on the price of anarchy arbitrarily close to k. For an upper bound suppose there is a Nash equilibrium with cost larger than $kc(\mathcal{S}^*)$. Then at least one player pays at least the cost $c(\mathcal{S}^*)$ and can thus deviate to purchase $\mathcal{S}^*$ completely by herself. This contradicts the assumption of a Nash equilibrium. The argumentation allows to show a price of anarchy of exactly k even for non-metric games. To derive a bound on the price of stability, we note that there are games without Nash equilibria.

Lemma 1. *There is a metric UFL game without Nash equilibria.*

Consider the game in Fig. 1(a). We assume that $c(f_1) = c(f_3) = 1$ and $c(f_2) = 1.5$. Player 1 either contributes to f_2 or to f_1 and f_3. If she purchases only $c(f_2)$, for it is best for player 2 to open one other facility, e.g. f_1. In this case it is better for player 1 to connect to f_1 and pay for opening f_3 as well. Then player 2 can drop f_1 and simply connect to f_3. This will create an incentive for player 1 to return to paying only for

f_2. Although this is not a formal proof, it illustrates the cycling objectives inherent in the game. In a Nash equilibrium each terminal must be connected to an opened facility. Thus, formally all Nash equilibria can be considered by seven cases – depending on the different sets of opened facilities. It can be shown that for each set of opened facilities the costs cannot be purchased by a Nash equilibrium payment scheme. This game and the game outlined for the lower bound on the price of anarchy can be combined to a class of games that yields a price of stability of $k - 2$. The construction is shown in Fig. 1(b). In addition, deciding the existence of Nash equilibria is NP-hard.

Theorem 1. *The price of stability in the metric UFL game is at least* $k - 2$.

Theorem 2. *It is NP-hard to decide whether a metric UFL game has a Nash equilibrium.*

Both results extend easily to non-metric games. Thus, exact Nash equilibria can be quite costly and hard to compute. For some classes of games, however, there is a cheap Nash equilibrium. In particular, results in [2] can be used to show that UFL games with a single terminal per player allow for an iterative improvement procedure that improves both stability and approximation ratio. The price of stability is 1, and $(1 + \epsilon, 1.52)$-approximate Nash equilibria can be found using a recent approximation algorithm [16] to compute a starting solution. In addition, we show that there is another class of games with cheap equilibria, which can be computed efficiently.

Theorem 3. *For any metric UFL game, in which the underlying UFL problem has integrality gap 1, the price of stability is 1. An optimal Nash equilibrium can be computed in polynomial time.*

The payments are determined as follows. Reconsider the IP formulation (1) and its corresponding LP-relaxation obtained by allowing $y_f, x_{tf} \geq 0$. The integrality gap is assumed to be 1, so the optimum solution (x^*, y^*) to (1) is optimal for the relaxation. Using the optimum solution (γ^*, δ^*) to the dual of the LP-relaxation we assign $p_i^o(f) = y_f^* \left(\sum_{t \in T_i} \delta_{tf}^*\right)$ for player i and each facility f. In addition, we let player i contribute $p_i^c(t, f) = x_{tf}^*(\gamma_t^* - \delta_{tf}^*)$ for each $t \in T_i$ and $f \in F$. The argument that this gives a Nash equilibrium relies on LP duality and complementary slackness.

For general games we consider approximate Nash equilibria. This concept is motivated by the assumption that stability evolves if each player has no *significant* incentive to deviate. Formally, for (α, β)-approximate Nash equilibria the stability ratio $\alpha \geq 1$ specifies the violation of the Nash equilibrium inequality, and $\beta \geq 1$ is the approximation ratio of the social cost.

Theorem 4. *For the metric UFL game there is a primal-dual algorithm to derive* $(3, 3)$*-approximate Nash equilibria in polynomial time.*

Proof. In Algorithm 1 we denote a terminal by t, a facility by f, and the player owning t by i_t. The algorithm raises budgets for each terminal, which are offered for purchasing the connection and opening costs. Facilities are opened if the opening costs are covered by the total budget offered, and if they are located sufficiently far away from other opened facilities. For the approximation ratio of 3 we note that the algorithm is a primal-dual method for the UFL problem [17, 19].

For the analysis of the stability ratio consider a single player i and her payments. Note that the algorithm stops raising the budget of a terminal by the time it becomes directly or indirectly connected. We will first show that for the final budgets $\sum_{t \in T_i} B_t$ is a lower bound on the cost of any deviation for player i. For any terminal t we denote by $f(t)$ the facility t is connected to in the calculated solution. Verify that $c(t, f) \geq B_t$ for any terminal t and any opened facility $f \neq f(t)$. Hence, if a player has a deviation that improves upon B_t, it must open a new facility and connect some of her terminals to it. By opening a new facility, however, the player is completely independent of the cost contributions of other players. Similar to [19] we can argue that the final budgets yield a feasible solution for the dual of the LP-relaxation. Hence, they form a 3-approximately budget a balanced core solution for the cooperative game [13]. Now suppose there is a deviation for a player, which opens a new facility f and connects a subset of her terminals T_f to f thereby improving upon the budgets. Then the cost of $c(f) + \sum_{t \in T_f} < \sum_{t \in T_f} B_t$. This, however, would mean that the coalition formed by T_f in the coalitional game has a way to improve upon their budgets, which is a contradiction to B_t having the core-property. Hence, we know that $\sum_{t \in T_i} B_t$ is a lower bound on every deviation cost. Finally, note that for every directly connected terminal $t \in T_i$ player i pays B_t. A terminal t becomes indirectly connected only if it is unconnected and tight to a facility f by the time f is definitely closed. f becomes definitely closed only if there is another previously opened facility f' at distance $2B_t$ from f. Hence, there is an edge $c(t, f') \leq 3B_t$ by the metric inequality. So in the end player i pays at most $3B_t$ when connecting an indirectly connected terminal to the closest opened facility. This establishes the bound on the stability ratio. □

In terms of lower bounds there is no polynomial time algorithm with an approximation ratio of 1.463 unless $NP \subset DTIME(n^{O(\log \log n)})$ [9]. The following theorem shows that in the game of Fig. 1(a) the cost of any feasible solution cannot be distributed to get an approximate Nash equilibrium with stability ratio $\alpha \leq 1.097$.

Theorem 5. *There is a metric UFL game in which for every (α, β)-approximate Nash equilibrium $\alpha > 1.097$.*

Relation to cooperative games In the cooperative game each terminal is a single player. The foremost stability concept is the core – the set of cost allocations assigning any coalition of players at most the cost of the optimum solution for this coalition only. A Nash equilibrium in our game guarantees this property only for the coalitions represented by our players. On the other hand the investments of a player now alter the cost of optimal solutions for other players. This feature makes overcovering the central problem that needs to be resolved to provide cheap solutions with low incentives to deviate. For deriving cheap approximately budget balanced core solutions the method of dual fitting can be applied, which scales the assigned payments of players to dual feasibility. The scaling factor then yields a factor for competitiveness, the notion in cooperative games analog to the stability ratio. In our non-cooperative framework the same simple scaling unfortunately does not work. In particular, for recently proposed greedy methods with better approximation ratios the factor for the approximation ratio does not translate.

Algorithm 1: Primal-dual algorithm for (3,3)-approximate Nash equilibria

In the beginning all terminals are unconnected, all budgets B_t are 0, and all facilities closed. Raise budgets of *unconnected* terminals at the same rate until one of the following events occurs. We denote the current budget of unconnected terminals by B. We call a terminal t *tight* with facility f if $B_t \geq c(t, f)$.

1. An unconnected terminal t goes tight with an opened facility f.
   In this case set t *connected* to f and assign player i_t to pay $p^c_{i_t}(t, f) = c(t, f)$.
2. For a facility f not yet definitely closed the sum of the budgets of unconnected and indirectly connected terminals t pays for opening and connection costs:
   $\sum_t \max(B_t - c(t, f), 0) = c(f)$. Then stop raising the budgets of the unconnected tight terminals. Also,
   (a) if there are opened facility f' and terminal t' with $c(t', f) + c(t', f') \leq 2B$, set f *definitely closed* and all unconnected terminals t tight with f *indirectly connected*.
   (b) Otherwise open f and set all terminals *directly connected* to f, which are tight with f and not yet directly connected to some other facility. For each such terminal assign player i_t to pay $p^c_{i_t}(t, f) = c(t, f)$ and $p^o_{i_t}(f) = B_t - c(t, f)$.

In the end connect all indirectly connected terminals to the closest opened facility and assign the corresponding players to pay for the connection cost.

Lemma 2. *The payments computed by recent greedy algorithms [12, 16] yield a stability ratio of $\Omega(k)$.*

2.2 Extensions

Connection-Restricted Facility Location Games We extend the game from UFL to connection-restricted facility location (CRFL) problems as considered in [8]. Instead of the constraints $y_f - x_{tf} \geq 0$ there is for each facility f a set of feasible subsets of terminals that can be connected simultaneously to f. This formulation allows for instance capacity, quota, or incompatibility constraints and thus encompasses several well-known generalizations of the problem. For subclasses of these games some of the previous results can be extended to hold. Details are deferred to the full version.

Theorem 6. *For complete CRFL games, in which a partially conic relaxation of the underlying CRFL problem has integrality gap 1, the price of stability is 1.*

Single source connection games By appropriately changing opening and connection costs most of the previous results translate in some reduced form to the SSC game with any number of terminals per player. As the previous algorithms in [2, 10] construct approximate Nash equilibria purchasing $\mathcal{S}^*$, we explicitly examine a lower bound for this case.

Corollary 1. *There is a SSC game, in which for every (α, β)-approximate Nash equilibrium $\alpha > 1.0719$. For approximate Nash equilibria with $\beta = 1$ purchasing $\mathcal{S}^*$ the bound increases to $\alpha > 1.1835$.*

Corollary 2. *In the SSC game the price of stability is at least $k-2$.*

Corollary 3. *It is NP-hard to decide whether a SSC game has a Nash equilibrium.*

3 Covering Games

Covering games and their equilibria are defined similarly to the facility location case. A covering integer problem (CIP) is given as

$$\begin{array}{lll} \text{Min} & \sum_{f=1}^{n} c(f)x_f & \\ \text{subject to} & \sum_{f=1}^{n} a(t,f)x_f \geq b(t) & \text{for all } t=1,\ldots,m \\ & x_f \in \mathbb{N} & \text{for all } f=1,\ldots,n. \end{array} \tag{2}$$

All constants are assumed to have non-negative (rational) entries $a(t,f), b(t), c(f) \geq 0$ for all $t = 1,\ldots,m$ and $f = 1,\ldots,n$. Associated with each of the k non-cooperative players is a subset of the constraints C_i, which she strives to satisfy. Integral units of a resource f have cost $c(f)$. They must be bought to be available for constraint satisfaction. Each player i chooses as a strategy a *payment function* $p_i : \{1,\ldots,n\} \to \mathbb{R}^n_+$, which specifies her non-negative contribution to each resource f. Then an integral number of x_f units of resource f are considered *bought* if x_f is the largest integer such that $\sum_i p_i(f) \geq c(f)x_f$. A bought unit can be used by all players for constraint satisfaction – no matter whether they contribute or not. We assume that if player i offers some amount $p_i(f)$ to resource f, and x_f units are bought in total, then her contribution to each unit is $p_i(f)/x_f$. Each player strives to minimize her cost, but insists on satisfying her constraints. We can translate definitions of exact and approximate Nash equilibria in this game directly from the UFL game. In addition, observations similar to the ones made in Sect. 2 can be used to simplify a game. Hence, in the following we will assume w.l.o.g. that the constraint sets C_i of the players form a partition of the constraints of the CIP. Note that in a Nash equilibrium no player contributes to an unbought unit, so the equality $\sum_i p_i(f) = c(f)x_f$ holds.
In the covering game prices of anarchy and stability behave similarly as in the metric UFL game. Using the results for vertex cover games in [3] and similar observations for the price of anarchy as in Sect. 2.1, we can see that the price of anarchy in the covering game is exactly k and the price of stability is at least $k-1$. Furthermore, even for vertex cover games it is NP-hard to decide, whether a covering game has a Nash equilibrium. Hence, we again focus on classes of games, for which cheap Nash equilibria exist. For variants of set cover games we have the following results. $\mathcal{F}$ denotes the maximum frequency of any element in the sets.

Theorem 7. *If for a set cover game, the integrality gap of the CIP is 1, the price of stability is 1 and an optimal Nash equilibrium can be found in polynomial time.*

Theorem 8. *The price of stability in singleton set multi-cover games is 1.*

Theorem 9. *There is a primal-dual algorithm to compute a $(\mathcal{F},\mathcal{F})$-approximate Nash equilibrium for set cover games.*

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