Earning Limits in Fisher Markets with Spending-Constraint Utilities

Xiaohui Bei[1], Jugal Garg[2], Martin Hoefer[3], Kurt Mehlhorn[4]

[1] Nanyang Technological University, Singapore. xhbei@ntu.edu.sg
[2] University of Illinois at Urbana-Champaign, USA. jugal@illinois.edu
[3] Goethe University Frankfurt, Germany. mhoefer@cs.uni-frankfurt.de
[4] MPI Informatik, Germany. mehlhorn@mpi-inf.mpg.de

Abstract. Earning limits are an interesting novel aspect in the classic Fisher market model. Here sellers have bounds on their income and can decide to lower the supply they bring to the market if income exceeds the limit. Beyond several applications, in which earning limits are natural, equilibria of such markets are a central concept in the allocation of indivisible items to maximize Nash social welfare.

In this paper, we analyze earning limits in Fisher markets with linear and spending-constraint utilities. We show a variety of structural and computational results about market equilibria. The equilibrium price vectors form a lattice, and the spending of buyers is unique in non-degenerate markets. We provide a scaling-based algorithm that computes an equilibrium in time $O(n^3\ell\log(\ell+nU))$, where n is the number of agents, $\ell \geq n$ a bound on the segments in the utility functions, and U the largest integer in the market representation. Moreover, we show how to refine any equilibrium in polynomial time to one with minimal prices, or one with maximal prices (if it exists). Finally, we discuss how our algorithm can be used to obtain in polynomial time a 2-approximation for Nash social welfare in multi-unit markets with indivisible items that come in multiple copies.

1 Introduction

Fisher markets are a fundamental model to study competitive allocation of goods among rational agents. In a Fisher market, there is a set B of buyers and a set G of divisible goods. Each buyer $i \in B$ has a budget $m_i > 0$ of money and a utility function u_i that maps any bundle of goods to a non-negative utility value. Each good $j \in G$ is assumed to come in unit supply and to be sold by a separate seller. A *competitive* or *market equilibrium* is an allocation vector of goods and a vector of prices, such that (1) every buyer spends his budget to buy an optimal bundle of goods, and (2) supply equals demand.

Fisher markets have been studied intensively in algorithmic game theory. For many strictly increasing and concave utility functions, market equilibria can be described by convex programs [12, 17]. There are a variety of algorithms for computing market equilibria [9, 10, 14, 21]. For linear markets, there are even algorithms that run in strongly polynomial time [16, 20]. Moreover, simple tâtonnement [5, 7] or proportional response dynamics [4, 22] converge to equilibrium (quickly).

A common assumption in all this work is that utility functions are non-satiated, that is, the utility of every buyer i strictly increases with amount of good allocated, and the utility of every seller j strictly increases with the money earned. Consequently, when buyers and sellers are price-taking agents, it is in their best interest to spend their entire budget and bring all supply to the market, resp. In this paper, we study new variants of linear Fisher markets with satiated utility functions recently proposed in [6]. In these markets, each seller has an earning limit, which gives him an incentive to possibly reduce the supply that he brings to the market. This is a natural property in many domains, e.g., when sellers have revenue targets. Many properties of such markets are not well-understood.

Interestingly, equilibria in Fisher markets with earning limits also relate closely to *fair allocations of indivisible* items. There has been a surge of interest in allocating indivisible items to maximize Nash social welfare. Very recent work [1, 8] has provided the first constant-factor approximation algorithms for this important problem. The algorithms first compute and then cleverly round a market equilibrium of a Fisher market with earning limits. The tools and techniques for computing market equilibria are a key component in this approach.

In this paper, we consider algorithmic and structural properties of markets with earning limits and spending-constraint utilities. Spending-constraint utilities are a natural generalization of linear utilities with many additional applications [10, 18]. We show structural properties of equilibria and provide new and improved polynomial-time algorithms for computation. Moreover, we show how these algorithms can be used to approximate Nash social welfare in markets where each item j is provided in d_j copies (where d_j is a given integer). We obtain the first polynomial-time approximation algorithms for multi-unit markets.

Contribution and Outline After formal discussion of the market model, we discuss some preliminaries in Section 2, including a formal condition for existence of equilibrium. In Section 3, we show that the set of equilibrium price vectors forms a lattice. While there always exists an equilibrium with pointwise smallest prices, an equilibrium with largest prices might not exist. Moreover, in non-degenerate markets (for a formal definition see Section 2) the spending of buyers in every equilibrium is unique.

In Section 4 we outline a novel algorithm to compute an equilibrium in time $O(n^3\ell\log(\ell + nU))$, where n is the total number of agents, ℓ is the maximum number of segments in the description of the utility functions that is incident to any buyer or any good, and U is the largest integer in the representation of utilities, budgets, and earning limits. For linear markets, the running time simplifies to $O(n^4 \log nU))$. Our algorithm uses a scaling technique with decreasing prices and maintains assignments in which buyers overspend their money. A technical challenge is to maintain rounded versions of the spending restrictions in the utility functions. The algorithm runs until the maximum overspending of all buyers becomes tiny and then rounds the outcome to an exact equilibrium. Given an arbitrary equilibrium, we show how to find in polynomial time an equilibrium with smallest prices, or one with largest prices (if it exists).

Finally, in Section 5 we round a market equilibrium in linear markets with earning caps to an allocation in indivisible multi-unit markets to approximate the Nash social welfare. In these markets, the representation is given by the set of items and for each item j a number d_j of the available copies. The direct application of existing algorithms [1, 8] would require pseudo-polynomial time. Instead, we show how to adjust the rounding procedure in [8] to run in strongly polynomial time. The resulting algorithm yields a 2-approximation and runs in time $O(n^4 \log(nU))$, which is polynomial in the input size.

Related Work For Fisher markets we focus on some directly related work about computation of market equilibria. For markets with linear utilities a number of polynomial-time algorithms have been derived [9, 14, 21], including ones that run in strongly polynomial time [16, 20]. For spending-constraint utilities in exchange markets [10] a polynomial-time algorithm was recently obtained [2]. For Fisher markets with spending-constraint utilities, the algorithm by Vegh [20] runs in strongly polynomial time.

Linear markets with either utility or earning limits were studied only recently [3, 6]. The equilibria solve standard convex programs. The Shmyrev program for earning limits also applies to spending-constraint utilities. Our paper complements our previous results [3] on linear markets with utility limits, where we proved that (1) equilibria form a lattice, (2) an equilibrium with maximum prices can be computed in time $O(n^8 \log(nU))$, (3) it can be refined in polynomial time to an equilibrium with minimum prices, and (4) several related problem variants are NP- or PPAD-hard. The framework of [19] provides an (arbitrary) equilibrium in time $O(n^5 \log(nU))$. For earning limits, our algorithm runs in time $O(n^3 \ell \log(\ell + nU))$ for spending-constraint and $O(n^4 \log(nU))$ for linear utilities. It computes an approximate solution that can be rounded to an exact equilibrium. An approximate solution could also be obtained with classic algorithms for separable convex optimization [13, 15]. These algorithms have slower running times – in particular, the algorithm of [15] obtains the required precision only in time $O(n^3 \ell^2 \log(\ell) \log(\ell + nU))$.

An interesting open problem are strongly polynomial-time algorithms for arbitrary earning limits. A non-trivial challenge in adjusting [16] is the precision of intermediate prices. For the framework of [20] the challenge lies in generalizing the Error-method to markets with earning limits.

Approximating optimal allocations of indivisible items that maximize Nash social welfare has been studied recently for markets with additive [6, 8] and separable concave valuations [1]. Here equilibria of markets with earning limits can be rounded to yield a 2-approximation. We extend this approach to markets with multi-unit items, where each item j comes in d_j copies (and the input includes d_j in standard logarithmic coding). In contrast to the direct, pseudo-polynomial extensions of previous work, we show how to obtain a 2-approximation in polynomial time.

2 Preliminaries

In a spending-constraint Fisher market with earning limits, there is a set B of buyers and a set G of goods. Every buyer $i \in B$ has a *budget* $m_i > 0$ of money. The utility of buyer i is a *spending-constraint function* given by non-empty sets of *segments* $K_{ij} = \{(i,j,k) \mid 1 \leq k \leq \ell_{ij}\}$ for each good $j \in G$. Each segment $(i,j,k) \in K_{ij}$ comes with a *utility value* u_{ijk} and a *spending limit* $c_{ijk} > 0$. We assume that the utility function is piecewise linear and concave, i.e., $u_{ijk} > u_{ij,k+1} > 0$ for all $\ell_{ij} - 2 \geq k \geq 1$. W.l.o.g. we assume that the last segment has $u_{ij\ell_{ij}} = 0$ and $c_{ijk} = \infty$.

Buyer i can spend at most an amount of c_{ijk} of money on segment (i,j,k). We use $\mathbf{f} = (f_{ijk})_{(i,j,k) \in K_{ij}}$ to denote the spending of money on segments. $\mathbf{f}$ is termed *money flow*. A segment is *closed* if $f_{ijk} \geq c_{ijk}$, otherwise *open*. For notational convenience, we let $f_{ij} = \sum_{(i,j,k) \in K_{ij}} f_{ijk}$.

Given a vector $\mathbf{p} = (p_j)_{j \in G}$ of strictly positive *prices* for goods, a money flow results in an allocation $x_{ij} = \sum_k f_{ijk}/p_j$ of good j. The *bang-per-buck ratio* of segment (i,j,k) is $\alpha_{ijk} = u_{ijk}/p_j$. To maximize his utility, buyer i spends his budget m_i on segments in non-increasing order of bang-per-buck ratio, while respecting the spending limits. A bundle $\mathbf{x}_i = (x_{ij})_{j \in G}$ that results from this approach is termed a *demand bundle* and denoted by $\mathbf{x}_i^*$. The corresponding money flow on the segments is termed *demand flow* $\mathbf{f}_i^*$.

Demand bundles and flows might not be unique, but they differ only on the allocated segments with smallest bang-per-buck ratio. This smallest ratio is termed *maximum bang-per-back (MBB) ratio* and denoted by α_i. Note that α_i is unique given $\mathbf{p}$. All segments with $\alpha_{ijk} \geq \alpha_i$ are termed *MBB segments*. The segments with $\alpha_{ijk} = \alpha_i$ are termed *active segments*. We assume w.l.o.g. $m_i \leq \sum_{j,k: u_{ijk} > 0} c_{ijk}$, since no buyer would spend more, and we can assume there is no allocation on segments with $u_{ijk} = 0$. Therefore, we assume buyers always spend all their money.

In this paper, we study a natural condition on seller supplies. Each good is owned by a different seller, and the seller has a maximum endowment of 1. Seller j comes with an *earning limit* d_j. He only brings a supply $e_j \leq 1$ that suffices to reach this earning limit under the given prices. Intuitively, while each seller has utility $\min\{d_j, e_j p_j\}$, we also assume that he has tiny utility for unsold parts of his good. Hence, he only brings a supply to earn d_j.

More formally, the *active price* of good j is given by $p_j^a = \min(d_j, p_j)$. His good is *capped* if $p_j^a = d_j$ and *uncapped* otherwise. A *thrifty supply* is $e_j = p_j^a/p_j$, which guarantees $e_j p_j \leq d_j$, i.e., the earning limit holds when market clears.

The goal is to find a *thrifty equilibrium.*

Definition 1. *A pair* $(\mathbf{f}, \mathbf{p})$ *is a* thrifty equilibrium *if (1)* $\mathbf{f}_i$ *is a demand flow for prices* $\mathbf{p}$ *for every* $i \in B$*, and (2)* $\sum_{i,k} f_{ijk} = p_j^a$*, for every* $j \in G$*.*

Proposition 1. *Across all thrifty equilibria: (1) the seller incomes are unique; (2) there is a unique set of uncapped goods, and their prices are unique; and (3) uncapped goods are available in full supply, capped goods in thrifty supply.*

These uniqueness properties are a direct consequence of the fact [6] that thrifty equilibria are the solutions of the following convex program.

$$
\begin{aligned}
\text{Max. } & \sum_{i\in B}\sum_{j\in G}\sum_{(i,j,k)\in K_{ij}} f_{ijk}\log u_{ijk} - \sum_{j\in G}(q_j \log q_j - q_j) \\
\text{s.t. } & \sum_{j\in G}\sum_{(i,j,k)\in K_{ij}} f_{ijk} = m_i && \forall i \in B \\
& \sum_{i\in B}\sum_{(i,j,k)\in K_{ij}} f_{ijk} = q_j && \forall j \in G \\
& f_{ijk} \le c_{ijk} && \forall (i,j,k) \in K_{ij} \\
& q_j \le d_j && \forall j \in G \\
& f_{ijk} \ge 0 && \forall i \in B,\ j \in G, (i,j,k) \in K_{ij}
\end{aligned}
\tag{1}
$$

The incomes of sellers and, consequently, the sets of capped and uncapped goods are unique in all thrifty equilibria. The money flow, allocation, and prices of capped goods might not be unique.

Buyers always spend all their budget, but this can be impossible when every seller must not earn more than its limit[5]. Then a thrifty equilibrium does not exist. This, however, turns out to be the only obstruction to nonexistence.

Let $\hat{B} \subseteq B$ be a set of buyers, and $N(\hat{B}) = \{j \in G \mid u_{ij1} > 0, i \in \hat{B}\}$ be the set of goods such that there is at least one buyer in $\hat{B}$ with positive utility on its first segment for the good. The following *money clearing* condition states that buyers can spend their money without violating the earning limits.

Definition 2 (Money Clearing). *A market is* money clearing *if for every subset of buyers $\hat{B} \subseteq B$ there is a flow f such that*

$$
\begin{aligned}
& f_{ij} \le \sum_{k=1}^{k^+} c_{ijk},\ \forall i \in \hat{B}, \forall j \in N(\hat{B}), k^+ = \max\{k \mid u_{ijk} > 0\} \\
& \sum_{i\in\hat{B}} f_{ij} \le d_j, \qquad \forall j \in N(\hat{B}) \quad \textit{and} \quad \sum_{j \in N(\hat{B})} f_{ij} \ge m_i, \qquad \forall i \in \hat{B}\ .
\end{aligned}
\tag{MC}
$$

Money clearing is clearly necessary for the existence of a thrifty equilibrium. It is also sufficient since, e.g., our algorithm in Section 4 will successfully compute an equilibrium iff money clearing holds. Alternatively, it can be verified that this is the unique necessary and sufficient feasibility condition for convex program (1). It is easy to check condition (MC) by a max-flow computation. We therefore assume that our market instance satisfies it.

Lemma 1. *A thrifty equilibrium exists iff the market is money clearing.*

Let us define some more useful concepts for the analysis. For any pair $(\mathbf{f}, \mathbf{p})$ the *surplus of buyer i* is given by $s(i) = \sum_{j\in G} f_{ij} - m_i$, and the *surplus of good*

[5] Consider the example of a linear market with one buyer and one good. The utility is $u_{11} > 0$, the buyer has a budget $m_1 = 2$, the good has an earning limit $d_1 = 1$.

j is $s(j) = p_j^a - \sum_{i \in B} f_{ij}$. The *active-segment graph* $G(\mathbf{p})$ is a bipartite graph $(B \cup G, E)$ which contains edge $\{i, j\}$ iff there is some active segment (i, j, k). Note that there can be at most one active segment (i, j, k) for an (i, j). A market is called *non-degenerate* if the active segment graph for any non-zero $\mathbf{p}$ is a forest.

3 Structure of Thrifty Equilibria

Some Intuition. We start by providing some intuition for the structural results in the case where all utility functions are linear, i.e., with a single segment in every K_{ij}. Consider a thrifty equilibrium $(\mathbf{f}, \mathbf{p})$. Call an edge (i, j) $\mathbf{p}$-MBB if $u_{ij}/p_j = \alpha_i$. The active-segment graph here simplifies to an *MBB graph* $G(\mathbf{p})$.

Let C be any connected component of the MBB graph. The buyers in C spend all budget on the goods in C, and no other buyer spends money on the goods in C. Thus

$$\sum_{i \in C \cap B} m_i = \sum_{j \in C \cap G} p_j^a = \sum_{j \in C \cap G_u} p_j + \sum_{j \in C \cap G_c} d_j,$$

where G_c and G_u are the sets of capped and uncapped goods, resp. First, assume all goods in C are capped. Let r be a positive real and consider the pair $(\mathbf{f}, \mathbf{p}')$, where $p'_j = r \cdot p_j$ if $j \in C \cap G_c$ and $p'_j = p_j$ otherwise.

Note that the allocations for any good $j \in C \cap G_c$ are scaled by $1/r$. The pair $(\mathbf{f}, \mathbf{p}')$ is an equilibrium provided that all edges with positive allocation are also $\mathbf{p}'$-MBB and $p'_j \geq d_j$ for all $j \in C \cap G_c$. This certainly holds for $r > 1$ and $r - 1$ sufficiently small. If $p_j > d_j$ for all $j \in C$ this also holds for $r < 1$ and $1 - r$ sufficiently small. Thus, there is some freedom in choosing the prices in components containing only capped goods even for a fixed MBB graph. For non-degenerate instances, the money flow is unique (but not the allocation).

Now assume that there is at least one uncapped good in C, and let j_u be such an uncapped good. The price of any other good j in the component is linearly related to the price j_u, i.e., $p_j = \gamma_j p_{j_u}$, where γ_j is a rational number whose numerator and denominator is a product of utilities. Thus,

$$\sum_{i \in C \cap B} m_i = \sum_{j \in C \cap G} p_j^a = \sum_{j \in C \cap G_u} \gamma_j p_{j_u} + \sum_{j \in C \cap G_c} d_j,$$

and the reference price is uniquely determined. All prices in the component are uniquely determined. For a non-degenerate instance the money flow and allocation are also uniquely determined.

Suppose in a component C containing only capped goods we increase the prices by a common factor $r > 1$. We raise r continuously until a new MBB edge arises. If we can raise r indefinitely, no buyer in the component is interested in any good outside the component. Otherwise, a new MBB edge arises, and then C is united with some other component. At this moment, the money flow over the new MBB edge is zero. If the newly formed component contains an uncapped good, prices in the component are fixed and money flow is exactly as in the moment of joining the components. Otherwise, we raise all prices in the newly formed component, and so on. If the market is non-degenerate, then money flow is unique, and money will never flow on the new MBB edge.

If the component contains only capped goods j with $p_j > d_j$, we can decrease prices continuously by a common factor $r < 1$ until a new MBB edge arises. If no MBB edge ever arises, no buyer outside the component is interested in any good in the component, which allows to argue as above.

We have so far described how the prices in a component of the MBB graph of an equilibrium are determined if at least one good is uncapped, and how the prices can be scaled by a common factor if all goods are capped. We have also discussed how components are merged and that the new MBB edge arising in a merge will never carry nonzero flow. Components can also be split if they contain an edge with zero flow.

Consider an equilibrium $(\mathbf{f}, \mathbf{p})$ and assume $f_{ij} = 0$ for some edge (i, j) of the MBB graph w.r.t. $\mathbf{p}$. Let C be the component containing (i, j) and let C_1 and C_2 be the components of $C \setminus \{i, j\}$. Let the instance be non-degenerate. Hence, the MBB graph is a forest. If we want to retain all MBB edges within C_1 and C_2 and only drop (i, j), we have to either increase all prices in the subcomponent containing j or decrease all prices in the subcomponent containing i. Both options are infeasible if both components contain a good with price strictly below its earning limit. The first option is feasible if the component containing j contains only goods with prices at least their earning limits. The latter option is feasible if the component containing i contains only goods with prices strictly larger than their earning limits. The split does not affect the money flow.

If the above described changes allow to change any equilibrium into any other equilibrium, then money flow should be unique across all equilibria. Moreover, the set of edges carrying flow should be the same in all equilibria. The MBB graph for an equilibrium contains these edges, and maybe some more edges that do not carry flow. Next, we prove that this intuition captures the truth, even for the general case of spending-constraint utility functions.

Lattice Structure. We characterize the set of price vectors of thrifty equilibria, which we denote by $\mathcal{P} = \{\mathbf{p} \mid \exists \mathbf{f} \text{ s.t. } (\mathbf{f}, \mathbf{p}) \text{ is a thrifty equilibrium}\}$. For money clearing markets, we establish two results: (1) the set of equilibrium price vectors forms a lattice, and (2) the money flow is unique in nondegenerate markets. For the first result, we consider the coordinate-wise comparison, i.e., $\mathbf{p} \geq \mathbf{p}'$ iff $p_j \geq p'_j, \forall j \in G$.

Theorem 1. *The pair $(\mathcal{P}, \geq)$ is a lattice.*

The proof relies on the following structural properties. Given $\mathbf{p}$ and $\mathbf{p}'$, we partition the set of goods into sets $S_r = \{j \mid p'_j = r \cdot p_j\}$, for $r > 0$. For a price vector $\mathbf{p}$, let segment (i, j, k) be $\mathbf{p}$-MBB if $u_{ijk}/p_j \geq \alpha_i$, and $\mathbf{p}$-active if $u_{ijk}/p_j = \alpha_i$. For a set T of goods and an equilibrium $(\mathbf{f}, \mathbf{p})$, let

$$K(T, \mathbf{p}) = \{(i, j, k) \mid \text{segment is } \mathbf{p}\text{-MBB for some } j \in T\},$$
$$K_a(T, \mathbf{p}) = \{(i, j, k) \mid f_{ijk} > 0 \text{ for some } j \in T \text{ and some equilibrium } (\mathbf{f}, \mathbf{p})\},$$

where the sets denote the set of $\mathbf{p}$-MBB segments for goods in T and the ones on which some good in T is allocated. Note that $K_a(T, \mathbf{p}) \subseteq K(T, \mathbf{p})$.

Lemma 2. *For any two thrifty equilibria $E = (\mathbf{f}, \mathbf{p})$ and $E' = (\mathbf{f}', \mathbf{p}')$:*

1. *$K_a(S_r, \mathbf{p}) = K_a(S_r, \mathbf{p}')$ for every $r > 0$, i.e., the goods in S_r are allocated on the same set of segments in both equilibria.*
2. *$K_a(S_r, \mathbf{p}) = K_a(S_r, \mathbf{p}') \subseteq K(S_r, \mathbf{p}') \subseteq K(S_r, \mathbf{p})$ for $r > 1$. Similarly, $K_a(S_r, \mathbf{p}') = K_a(S_r, \mathbf{p}) \subseteq K(S_r, \mathbf{p}) \subseteq K(S_r, \mathbf{p}')$ for $r < 1$. For $r = 1$, $K_a(S_r, \mathbf{p}') = K_a(S_r, \mathbf{p})$.*
3. *If $f_{ijk} > 0$ for $(i, j, k) \in K_a(S_r, \mathbf{p})$ with $r > 1$, then (i, j, k) is $\mathbf{p}'$-MBB. If $f'_{ijk} > 0$ for $(i, j, k) \in K_a(S_r, \mathbf{p}')$ with $r < 1$, then (i, j, k) is $\mathbf{p}$-MBB.*

Corollary 1. *There exists a thrifty equilibrium with coordinate-wise lowest prices. Among all thrifty equilibria, it yields the largest supply in the market and the maximum utility for every buyer.*

Theorem 2. *In a non-degenerate market, all thrifty equilibria have the same money flow.*

The convex program implies that there is a unique income for each seller. This is consistent with our observation that a good can have different prices in two equilibria only when income equals its earning limit.

While existence of an equilibrium with smallest prices is guaranteed, we might or might not have an equilibrium with coordinate-wise largest prices (e.g., when all goods are capped in equilibrium, prices can be raised indefinitely).

4 Algorithms to Compute Thrifty Equilibria

Scaling Algorithm. We first propose and discuss a polynomial-time scaling algorithm to compute a thrifty equilibrium. We begin with defining some useful tools and concepts. The *active-segment network* $N(\mathbf{p}) = (\{s, t\} \cup B \cup G, E)$ contains a node for each buyer and each good, along with two additional nodes s and t. It contains every edge (s, i) for $i \in B$ with capacity $m_i - c_i^c$, where $c_i^c = \sum_{(i,j,k) \text{ closed}} c_{ijk}$. Also, it contains every (j, t) for $j \in G$ with capacity $p_j^a - c_j^c$, where $c_j^c = \sum_{(i,j,k) \text{ closed}} c_{ijk}$. It contains edge (i, j) with infinite capacity iff there is some active segment (i, j, k). Finally, the *active-residual network* $G_r(\mathbf{f}, \mathbf{p})$ contains a node for each buyer and each good. It contains forward edge (i, j) iff there is some active segment (i, j, k) with $f_{ijk} < c_{ijk}$ and contains backward edge (j, i) iff there is some active segment (i, j, k) with $f_{ijk} > 0$. Moreover, $G_r(\mathbf{f}, \mathbf{p}, i)$ is the subgraph of $G_r(\mathbf{f}, \mathbf{p})$ induced by the set of all buyers $i' \in G_r(\mathbf{f}, \mathbf{p})$ such that there is an augmenting path from i' to i.

Our algorithm uses Δ-*discrete capacities* $\hat{c}_{ijk} = \lceil c_{ijk}/\Delta \rceil \cdot \Delta$ for all $i \in B, j \in G$ and $(i, j, k) \in K_{ij}$, where we iteratively decrease Δ. Initially, the algorithm overestimates the budget of buyer i, where it assumes the buyer has $r\Delta$ money and every segment has Δ-discrete capacities. Then $\mathbf{f}_i$ is a (Δ, r)-*discrete demand* for buyer i iff it is a demand flow for buyer i under these conditions.

We also adjust the definitions of MBB ratio, active segments, active-segment graph, network, and residual network to the case of Δ-discrete capacities. We denote these adjusted versions by $\hat{\alpha}$, $\hat{\mathcal{G}}(\mathbf{p})$, $\hat{N}(\mathbf{p})$, $\hat{\mathcal{G}}_r(\mathbf{f},\mathbf{p})$ and $\hat{\mathcal{G}}_r(\mathbf{f},\mathbf{p},j)$ resp.

Finally, we make a number of assumptions to simplify the stated bound on the running time. We assume w.l.o.g. that $|B| = |G|$ (by adding dummy buyers and/or goods) and define $n = |B| + |G|$. Moreover, we let $K_i = \bigcup_{j \in G} K_{ij}$ and $K_j = \bigcup_{i \in B} K_{ij}$ and assume w.l.o.g. that $\ell = |K_i| = |K_j| \geq n$ for every buyer i and every good j (by adding dummy segments with 0 utility).

Algorithm 1 computes a thrifty equilibrium in polynomial time. It uses descending prices and maintains a money flow on closed and open MBB segments with increasing precision and decreasing surplus. We call a run of the outer while-loop a Δ-*phase*. The algorithm runs until the precision parameter Δ is decreased to exponentially small size. Then a final rounding procedure PostProcessingS (described in Appendix A) rounds the solution to an exact equilibrium.

For the analysis, we use the following notion of Δ-feasible solution.

Definition 3. *Given a value $\Delta > 0$, a pair $(\mathbf{f},\mathbf{p})$ of flow and prices with $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{f} \geq \mathbf{0}$ is a Δ-*feasible solution *if*

- $\ell\Delta \leq s(i) \leq (\ell+1)\Delta, \forall i \in B$.
- *$\forall j \in G$: If $p_j < p_j^0$, then $0 \leq s(j) \leq \Delta$. If $p_j = p_j^0$, then $-\infty < s(j) \leq \Delta$.*
- *$\mathbf{f}$ is Δ-integral, and $f_{ijk} > 0$ only if (i,j,k) is a closed or open MBB segment w.r.t. Δ-discretized capacities.*

For the running time, note that prices are non-increasing. Once a capped good becomes uncapped, it remains uncapped. We refer to an execution of the repeat loop in Algorithm 1 as an iteration. After the initialization, there may be goods j for which d_j is smaller than the initial value of Δ and which receive flow from some buyer. As long as their surplus is negative, these goods keep their initial price. The following observations are useful to prove a bound on the running time. We also observe that the precision of prices and flow values is always bounded.

Lemma 3. *Once the surplus of a good is non-negative, it stays non-negative. If the surplus of a good is negative, its price is the initial price.*

Lemma 4. *The first run of the outer while-loop in Algorithm 1 takes $O(n^3\ell)$ time, every subsequent one takes $O(n^2\ell)$ time. At the end of each Δ-phase, the pair $(\mathbf{f},\mathbf{p})$ is a Δ-feasible solution.*

Lemma 5. *If all budgets, earning limits and utility values are integers bounded by U, then all flow values and prices at the end of each iteration are rational numbers whose denominators are at most poly$(1/\Delta, n, U^n)$.*

Finally, for correctness of the algorithm, it maintains the following condition resulting from (MC) for active prices.

Algorithm 1. Scaling Algorithm for Markets $\mathcal{M}^s$ with Earning Limits

Input : Fisher market $\mathcal{M}$ with spending constraint utilities and earning limits Budget m_i, earning limits d_j, and parameters u_{ijk}, c_{ijk}

Output : Thrifty equilibrium $(\mathbf{f}, \mathbf{p})$

$\Delta \leftarrow U^{n+1} \sum_{i \in B} m_i$; $p_j^0 \leftarrow n(\ell+1)\Delta, \forall j \in G$; $\mathbf{p} \leftarrow \mathbf{p}^0$
$f_i \leftarrow (\Delta, \ell+1)$-discrete demand for buyer i
while $\Delta > 1/(2\ell(2nU)^{4n})$ **do**
  $\Delta \leftarrow \Delta/2$;
  for *each closed segment* (i,j,k) **do** $f_{ijk} \leftarrow \lceil c_{ijk}/\Delta \rceil \cdot \Delta$
  for *each* $i \in B$ *with* $s(i) > (\ell+1)\Delta$ **do**
    Pick any active segment (i,j,k) with $f_{ijk} > 0$ and set $f_{ijk} \leftarrow f_{ijk} - \Delta$
  while *there is a good* j' *with* $s(j') > \Delta$ **do** // Δ-phase
    repeat // iteration
      $(\hat{B}, \hat{G}) \leftarrow$ Set of (buyers, goods) in $\hat{\mathcal{G}}_r(\mathbf{f}, \mathbf{p}, j')$
      $x \leftarrow 1$; Define $p_j \leftarrow x p_j, \forall j \in \hat{G}$ // active prices & surpluses change, too
      Decrease x continuously down from 1 until one of the following events occurs
        Event 1: $s(j') = \Delta$
        Event 2: $s(j) \le 0$ for a $j \in \hat{G}$
          $P \leftarrow$ path from j to j' in $\hat{\mathcal{G}}_r(\mathbf{f}, \mathbf{p}, j')$ // Δ-augmentation
          Update $\mathbf{f}$: $f_{ijk} = \begin{cases} f_{ijk} + \Delta \text{ if } (i,j) \text{ is a forward arc in } P \\ f_{ijk} - \Delta \text{ if } (i,j) \text{ is a backward arc in } P \\ f_{ijk} \quad \text{otherwise} \end{cases}$
        Event 3: A capped good becomes uncapped
        Event 4: New active segment (i,j,k) with $i \notin \hat{B}$, $j \in \hat{G}$, $f_{ijk} < \hat{c}_{ijk}$
    until *Event 1 or 2 occurs*
$(\mathbf{f}, \mathbf{p}) \leftarrow$ PostProcessingS$(\mathbf{f}, \mathbf{p})$ // see Appendix A

Lemma 6. *Let $\hat{B} \subseteq B$ be a set of buyers and let $N(\hat{B})$ be the goods having positive utility for some buyer in $\hat{B}$. At all times $\sum_{j \in N(\hat{B})} p_j^a - \sum_{i \in \hat{B}} m_i \ge 0$.*

Lemma 7. *Let $(\mathbf{f}, \mathbf{p})$ be the flow and price vector computed by the outer while-loop in Algorithm 1. The pair is Δ-feasible for $\Delta = 1/(2\ell(2nU)^{4n})$ and $-n(\ell+1)\Delta \le s(j) \le \Delta$ for all $j \in G$.*

Theorem 3. *Algorithm 1 computes a thrifty equilibrium for money-clearing markets $\mathcal{M}^s$ with earning limits in $O(n^3 \ell \log(\ell + nU))$ time.*

For the details on final rounding, we refer the reader to Appendix A.

Extremal Prices. Given an arbitrary thrifty equilibrium, Algorithm 2 computes a thrifty equilibrium with smallest prices. Algorithm 3 computes a thrifty equilibrium with largest prices if it exists. Otherwise, it yields a set S of goods for which prices can be raised indefinitely.

Algorithm 2. MinPrices

Input : Market parameters and any thrifty equilibrium $(\mathbf{f}, \mathbf{p})$
Output : Thrifty equilibrium with smallest prices

$E(\mathbf{f}) \leftarrow \{(i,j,k) \mid f_{ijk} > 0\}$; $G_c \leftarrow$ Set of capped goods at $(\mathbf{f}, \mathbf{p})$

Solve an LP in q_j and λ_i:
$$\begin{array}{ll} \min \sum_i \lambda_i & \\ q_j \leq u_{ijk}\lambda_i, & \text{for segment } (i,j,k) \in E(\mathbf{f}) \\ q_j = p_j, & \forall j \in G \setminus G_c \\ q_j \geq d_j, & \forall j \in G_c \\ \lambda_i, q_j \geq 0 & \forall i \in B, j \in G \end{array}$$

return $(\mathbf{f}, \mathbf{q})$

Algorithm 3. MaxPrices

Input : Market parameters and any thrifty equilibrium $(\mathbf{f}, \mathbf{p})$
Output : Thrifty equilibrium with largest prices

Initialize active price $p_j^a \leftarrow \min\{d_j, p_j\}$ for every good j
$S \leftarrow \{j \mid p_j > 0$ and j is not connected to any uncapped good in $G(\boldsymbol{p})\}$
while $S \neq \emptyset$ **do**
  $x \leftarrow 1$; Set prices $p_j \leftarrow xp_j, \ \forall j \in S$
  Increase x continuously from 1 until a new active segment appears
  Recompute S
return $(\mathbf{f}, \mathbf{p})$

Theorem 4. *Algorithm 2 computes a thrifty equilibrium with smallest prices.*

Theorem 5. *Algorithm 3 computes a thrifty equilibrium with largest prices if it exists.*

5 Nash Social Welfare in Additive Multi-Unit Markets

Using our algorithm to compute a thrifty equilibrium in linear markets with earning limits, we can approximate the optimal Nash social welfare for additive valuations, indivisible items, and multiple copies for each item. Here there are n agents and m items. For item j, there are $d_j \in \mathbb{N}$ copies. The valuation of agent i for an assignment x of goods is $v_i(x) = \sum_j v_{ij}x_{ij}$, where x_{ij} denotes the number of copies of item j that agent i receives. The goal is to find an assignment such that the Nash social welfare $\left(\prod_i v_i(x)\right)^{1/n}$ is maximized.

When all $d_j = 1$, the algorithm of [8] provides a 2-approximation [6]. It finds an equilibrium for a linear market, where each agent i is a buyer with $m_i = 1$, and each item j is a good with earning limit $d_j = 1$. Then it rounds the allocation to an integral assignment. The direct adjustment to handle $d_j \geq 1$ copies is to represent each copy of item j by a separate auxiliary item with unit supply (all valued exactly the same way as item j) and run the algorithm from [8]. A similar approach is used by [1] to provide a 2-approximation for separable concave utilities. This, however, yields a running time polynomial in $\max_j d_j$, which is only pseudo-polynomial for multi-unit markets (due to standard logarithmic coding of d_j's). We here outline a way to make the algorithm efficient.

Proposition 2. *There is a polynomial-time 2-approximation algorithm for maximizing Nash social welfare in multi-unit markets with additive valuations.*

References

1. N. Anari, T. Mai, S. O. Gharan, and V. Vazirani. Nash social welfare for indivisible items under separable, piecewise-linear concave utilities, 2016. CoRR abs/1612.05191.
2. X. Bei, J. Garg, and M. Hoefer. Ascending-price algorithms for unknown markets. In *Proc. 17th Conf. Economics and Computation (EC)*, page 699, 2016.
3. X. Bei, J. Garg, M. Hoefer, and K. Mehlhorn. Computing equilibria in markets with budget-additive utilities. In *Proc. 24th European Symp. Algorithms (ESA)*, pages 8:1–8:14, 2016.
4. B. Birnbaum, N. Devanur, and L. Xiao. Distributed algorithms via gradient descent for Fisher markets. In *Proc. 12th Conf. Electronic Commerce (EC)*, pages 127–136, 2011.
5. Y. K. Cheung, R. Cole, and N. Devanur. Tatonnement beyond gross substitutes? Gradient descent to the rescue. In *Proc. 45th Symp. Theory of Computing (STOC)*, pages 191–200, 2013.
6. R. Cole, N. Devanur, V. Gkatzelis, K. Jain, T. Mai, V. Vazirani, and S. Yazdanbod. Convex program duality, Fisher markets, and Nash social welfare. In *Proc. 18th Conf. Economics and Computation (EC)*, 2017. To appear.
7. R. Cole and L. Fleischer. Fast-converging tatonnement algorithms for one-time and ongoing market problems. In *Proc. 40th Symp. Theory of Computing (STOC)*, pages 315–324, 2008.
8. R. Cole and V. Gkatzelis. Approximating the Nash social welfare with indivisible items. In *Proc. 47th Symp. Theory of Computing (STOC)*, pages 371–380, 2015.
9. N. Devanur, C. Papadimitriou, A. Saberi, and V. Vazirani. Market equilibrium via a primal–dual algorithm for a convex program. *J. ACM*, 55(5), 2008.
10. N. Devanur and V. Vazirani. The spending constraint model for market equilibrium: Algorithmic, existence and uniqueness results. In *Proc. 36th Symp. Theory of Computing (STOC)*, pages 519–528, 2004.
11. R. Duan and K. Mehlhorn. A combinatorial polynomial algorithm for the linear Arrow-Debreu market. *Inf. Comput.*, 243:112–132, 2015.
12. E. Eisenberg and D. Gale. Consensus of subjective probabilities: The Pari-Mutuel method. *Ann. Math. Stat.*, 30(1):165–168, 1959.
13. D. Hochbaum and G. Shanthikumar. Convex separable optimization is not much harder than linear optimization. *J. ACM*, 37(4):843–862, 1990.
14. K. Jain. A polynomial time algorithm for computing the Arrow-Debreu market equilibrium for linear utilities. *SIAM J. Comput.*, 37(1):306–318, 2007.
15. A. Karzanov and T. McCormick. Polynomial methods for separable convex optimization in unimodular linear spaces with applications. *SIAM J. Comput.*, 26(4):1245–1275, 1997.
16. J. Orlin. Improved algorithms for computing Fisher's market clearing prices. In *Proc. 42nd Symp. Theory of Computing (STOC)*, pages 291–300, 2010.

17. V. Shmyrev. An algorithm for finding equilibrium in the linear exchange model with fixed budgets. *J. Appl. Indust. Math.*, 3(4):505–518, 2009.
18. V. Vazirani. Spending constraint utilities with applications to the adwords market. *Math. Oper. Res.*, 35(2):458–478, 2010.
19. L. Végh. Concave generalized flows with applications to market equilibria. *Math. Oper. Res.*, 39(2):573–596, 2014.
20. L. Végh. Strongly polynomial algorithm for a class of minimum-cost flow problems with separable convex objectives. *SIAM J. Comput.*, 45(5):1729–1761, 2016.
21. Y. Ye. A path to the Arrow-Debreu competitive market equilibrium. *Math. Prog.*, 111(1-2):315–348, 2008.
22. L. Zhang. Proportional response dynamics in the Fisher market. *Theoret. Comput. Sci.*, 412(24):2691–2698, 2011.

A Final Rounding

In this section, we describe the final rounding procedure `PostProcessingS` of Algorithm 1.

Algorithm 4. PostProcessingS($\mathbf{f}, \mathbf{p}$)

Input : ε-feasible solution $(\mathbf{f}, \mathbf{p})$ for $\varepsilon = 1/(2\ell(2nU)^{4n})$
Output : Market equilibrium $(\mathbf{f}', \mathbf{p}')$

$\hat{\mathcal{G}}(\mathbf{p}) = (B \cup G, E) \leftarrow$ active-segment graph at $\mathbf{p}$ w.r.t. Δ-discrete capacities $\hat{c}_{ijk}$
while $\exists$ *a component C in $\hat{\mathcal{G}}(\mathbf{p})$ s.t. all goods are capped* **do**
  $x \leftarrow 1$; Define prices as $p_j \leftarrow xp_j, \forall j \in C \cap G$
  Decrease x continuously down from 1 until one of the following events occurs
  Event 1: A capped good becomes uncapped
  Event 2: A new segment (i, j, k) becomes active `// components merge`
Recompute active-segment graph $\hat{\mathcal{G}}(\mathbf{p})$ and let $\mathcal{C}$ be the set of its components
Let K^c be the set of closed segments in $(\mathbf{f}, \mathbf{p})$ w.r.t. Δ-discrete capacities
$\hat{c}^c_j \leftarrow \sum_{(i,j,k)\in K^c} \hat{c}_{ijk}$ for every $j \in G$
$\hat{c}^c_i \leftarrow \sum_{(i,j,k)\in K^c} \hat{c}_{ijk}$ for every $i \in B$
for *each component $C \in \mathcal{C}$* **do**
  Set prices $\mathbf{p}$ as solution of the following system of equations
   (1) $u_{ijk}p_{j'} = u_{ij'k'}p_j$ (for active segments from a buyer i to goods j and j')
   (2) $\sum_{j\in C\cap G}(p^a_j - \hat{c}^c_j) - \sum_{i\in C\cap B}(m_i - \hat{c}^c_i) = \sum_{u\in C} s(u)$ (sum of surpluses)
  `//` $-n(\ell+1)\Delta \le s(u) \le (\ell+1)\Delta$
Let $\mathbf{A}\mathbf{p} = \mathbf{b}$ be the matrix form of the above system
Let $\mathbf{A}\mathbf{p}' = \mathbf{b}'$ be the system where $\mathbf{b}'$ is obtained from $\mathbf{b}$ after substituting $s(u) = 0$ and using c^c_u based on original c_{ijk}, for all $u \in B \cup G$
$\mathbf{f}' \leftarrow$ maximum s-t-flow in network $N(\mathbf{p}')$
return $(\mathbf{f}', \mathbf{p}')$

Note that by Lemma 7 we call PostProcessingS with a pair $(\mathbf{f}, \mathbf{p})$ that is Δ-feasible for $\Delta = 1/(2\ell(2nU)^{4n})$. Also $-n(\ell+1)\Delta \le s(u) \le (\ell+1)\Delta$ for every $u \in B \cup G$.

The while-loop in PostProcessingS ensures that all components of the active-segment graph $\hat{\mathcal{G}}(\mathbf{p})$ contain an uncapped good. For each component C of $\hat{\mathcal{G}}(\mathbf{p})$, the algorithm sets up a system of linear equations in price variables of the form $A\mathbf{p} = \mathbf{b}$, and we show that after perturbing $\mathbf{b}$ slightly, we get an equilibrium. Since we apply the same procedure on each component separately, we assume without loss of generality that there is exactly one component C of $\hat{\mathcal{G}}(\mathbf{p})$.

All goods in C are connected with each other through a set of active MBB edges. Whenever there are two active segments (i, j, k) and (i, j', k') for a buyer i and two goods j, j', we have the following relation between p_j and $p_{j'}$:

$$u_{ijk}p_{j'} = u_{ij'k'}p_j \tag{2}$$

It is easy to check that there are $|C \cap G| - 1$ of these MBB relations, which are linearly independent, and there is essentially one free price variable. Additionally, we have a condition for C on the sum of surpluses:

$$\sum_{j \in C \cap G} (p_j^a - \hat{c}_j^c) - \sum_{i \in C \cap B} (m_i - \hat{c}_i^c) = \sum_{u \in C} s(u) \ . \tag{3}$$

Since there is at least one uncapped good, the set of active prices p_j^a can be divided into a set for capped goods and a set for uncapped goods; $p_j^a = p_j$ for each uncapped j, and $p_j^a = d_j$ for each capped j. We can rewrite (3) as:

$$\sum_{j \in C \cap G \text{ uncapped}} p_j = \sum_{u \in C} s(u) - \sum_{j \in C \cap G \text{ capped}} d_j + \sum_{i \in C \cap B} m_i + \sum_{j \in C \cap G} \hat{c}_j^c - \sum_{i \in C \cap B} \hat{c}_i^c \ . \tag{4}$$

We can write the system of equations (2) and (4) in matrix form as $\mathbf{Ap} = \mathbf{b}$. All entries of $\mathbf{A}$ are integers due to our assumption on the input parameters, and $\mathbf{b}$ has exactly one non-zero entry resulting from (4). Now consider another system $\mathbf{Ap'} = \mathbf{b'}$ for a price vector $\mathbf{p'}$, where $\mathbf{b'}$ is obtained after setting $s(u) = 0$ and using c_u^c that sums the original capacities of closed segments. Next we show that $\mathbf{p'}$ gives an equilibrium. For this we show that there is a feasible flow in the active-segment network $N(\mathbf{p'})$ with min-cuts $(s, B \cup G \cup t)$ and $(s \cup B \cup G, t)$. The proof is based on an adaption of a similar result in [11].

Note that all entries of $\mathbf{A}$ are integers in $[-U, U]$. For $\mathbf{b'}$ all entries are integers in $[-2n\ell U, 2n\ell U]$. By Cramer's rule, the solution of $\mathbf{Ap'} = \mathbf{b'}$ is a vector of rational numbers with common denominator $D \leq (nU)^n$. That is, all p'_j are of form q_j/D, where both q_j and D are integers. Let $\varepsilon = n(\ell+1)\Delta + n\ell\Delta$. Since $||\mathbf{b}| - |\mathbf{b'}|| < \varepsilon$, we have $|p_j - p'_j| \leq \varepsilon D, \forall j$. Let $\varepsilon' = \varepsilon D^2$, then $|Dp_j - q_j| = |D(p_j - p'_j)| \leq \varepsilon D^2 = \varepsilon'$.

Lemma 8. *Every MBB segment with respect to* $\mathbf{p}$ *is also an MBB segment with respect to* $\mathbf{p'}$*. Furthermore, the set of capped and uncapped goods with respect to* $\mathbf{p}$ *and* $\mathbf{p'}$ *are the same.*

Proof. Suppose for two segments (i, j, k) and (i, j', k') we have $u_{ijk} p_{j'} \leq u_{ij'k'} p_j$, then

$$u_{ijk} q_{j'} \leq u_{ijk} D p_{j'} \leq D u_{ij'k'} p_j \leq D u_{ij'k'} (p'_j + \varepsilon D) \leq u_{ij'k'} q_j + \varepsilon' u_{ij'k'} < u_{ij'k'} q_j + 1.$$

Since both $u_{ijk} q_{j'}$ and $u_{ij'k'} q_j$ are integers, we have $u_{ijk} q_{j'} \leq u_{ij'k'} q_j$. This implies that all bang-per-buck relations for segments in the market are preserved. In particular, a segment is MBB w.r.t. $\mathbf{p}$ iff it is MBB w.r.t. $\mathbf{p'}$. The capped goods w.r.t. $\mathbf{p}$ remain capped w.r.t. $\mathbf{p'}$. Suppose $p_j \geq d_j$, then $q_j \geq Dp_j - \varepsilon' > Dd_j - 1$. Since q_j and d_j are integers, we have that $q_j \geq Dd_j$ and $p'_j \geq d_j$. Similarly, if $p_j \leq d_j$, then $q_j \leq Dp_j + \varepsilon' < Dd_j + 1$. Again, $q_j \leq Dd_j$ and $p'_j \leq d_j$. □

Note that after the rounding, all (active) prices $\mathbf{p'}$ are rational numbers with common denominator D. We assign to all closed segments the full amount c_{ijk}. For the active segments, consider the network $N(\mathbf{p'})$, and let c be the capacity of cut $(s, B \cup G \cup t)$ in $N(\mathbf{p}')$. Suppose there is a min-cut in $N(\mathbf{p}')$ with value less than c. Then that value is at most $c - 1/D$. This same cut in $\hat{N}(\mathbf{p})$ will have value at most $c - 1/D + \varepsilon D|G| + \varepsilon|B|$. Also the capacity of the cut $(s, B \cup G \cup t)$ in $\hat{N}(\mathbf{p})$ is at least $c - \varepsilon|B|$. Therefore the total surplus in $\hat{N}(\mathbf{p})$ is at least

$$c - \varepsilon|B| - (c - 1/D + \varepsilon D|G| + \varepsilon|B|) \geq 1/D - n\varepsilon D > \varepsilon,$$

which is a contradiction. Hence $(s, B \cup G \cup t)$ is a min-cut in $N(\mathbf{p}')$. Hence, after removing the money allocated to closed segments from buyer budgets and prices of goods, the remaining money on the active segments allows an allocation that clears the market. This shows that PostProcessingS works correctly. Hence, we know the algorithm is correct, requires only bounded precision and runs in polynomial time.

B Missing Proofs

B.1 Proof of Theorem 1

We first show that $(\overline{\mathbf{f}}, \overline{\mathbf{p}})$ is a thrifty equilibrium, where the spending $\overline{\mathbf{f}}$ is defined simply as $\overline{\mathbf{f}} = \mathbf{f}$. As such, the new state $(\overline{\mathbf{f}}, \overline{\mathbf{p}})$ is feasible with respect to earning limits and has thrifty supplies. It remains to show that the allocation is MBB. Compared to $\mathbf{p}$, $\overline{\mathbf{p}}$ has higher prices for the goods in S_r with $r > 1$. Hence the allocations to the goods in S_r with $r \leq 1$ are still MBB. Consider any good $j \in S_r$ with $r > 1$. If $f_{ijk} > 0$, then (i, j, k) is $\mathbf{p}'$-MBB by part 3 of Lemma 2. Thus $u_{ijk}/\overline{p}_j = u_{ijk}/p'_j = \alpha'_j \geq u_{ij'k'}/p'_{j'}$ for all $\mathbf{p}'$-active segments (i, j', k'). Since $p'_{j'} = \overline{p}_{j'}$ for $\ell \in S_r$ with $r > 1$ and $p'_{j'} \leq p_j = \overline{p}_j$ for $j' \in S_r$ with $r \leq 1$, we observe (i, j, k) is $\overline{\mathbf{p}}$-MBB. We conclude that $(\overline{\mathbf{f}}, \overline{\mathbf{p}})$ is a thrifty equilibrium.

Let us now consider $(\underline{\mathbf{f}}, \underline{\mathbf{p}})$ with spending $\underline{\mathbf{f}}$ defined as

$$\underline{f}_{ijk} = \begin{cases} f_{ijk} & \text{if } j \in S_r \text{ with } r > 1 \\ f'_{ijk} & \text{if } j \in S_r \text{ with } r \leq 1. \end{cases}$$

Again, the new state $(\underline{\mathbf{f}}, \underline{\mathbf{p}})$ is feasible with respect to earning limits and has thrifty supplies. It remains to show that the allocation is MBB.

Consider the goods in S_r with $r > 1$ and a buyer $i \in B_a(S_r, \mathbf{p})$. For prices $\mathbf{p}'$, we know by part 3 of Lemma 2 that for buyer i every segment (i, j, k) with $f_{ijk} > 0$ is $\mathbf{p}'$-MBB. Now, to reach $\underline{\mathbf{p}}$, we keep prices of S_r with $r \leq 1$ as in $\mathbf{p}'$ and decrease the prices of S_r with $r > 1$ to $\mathbf{p}$. As such, i does not obtain new MBB segments for goods in S_r with $r \leq 1$. For the remaining goods in S_r with $r \geq 1$, however, the allocation for i is MBB, since prices and spending for these goods are as in equilibrium $E = (\mathbf{f}, \mathbf{p})$.

Similarly, consider the goods in S_r with $r < 1$ and a buyer $i \in B_a(S_r, \mathbf{p}')$. For prices $\mathbf{p}$, we know by part 3 of Lemma 2 that for buyer i every segment (i, j, k) with $f'_{ijk} > 0$ is $\mathbf{p}$-MBB. Now, to reach $\underline{\mathbf{p}}$, we keep prices of S_r with $r \geq 1$ as in $\mathbf{p}$ and decrease the prices of S_r with $r < 1$ to $\mathbf{p}'$. As such, i does not obtain new MBB segments for goods in S_r with $r \geq 1$. For the remaining goods in S_r with

$r < 1$, however, the allocation for i is MBB, since prices and spending for these goods are as in equilibrium $E' = (\mathbf{f}', \mathbf{p}')$.

Finally, consider the goods in S_1 and a buyer $i \in B_a(S_r, \mathbf{p}') = B_a(S_r, \mathbf{p})$. Hence, since for $j \in S_1$ we have $p_j = p'_j$, a segment (i, j, k) is $\mathbf{p}$-MBB iff it is $\mathbf{p}'$-MBB. Repeating the above arguments for $r > 1$ and $r < 1$, we observe that for buyer i no new MBB segments evolve in S_r with $r \neq 1$. Hence, the spending $\underline{f}_{ij}$ is MBB for i.

We conclude that $(\underline{\mathbf{f}}, \underline{\mathbf{p}})$ is a thrifty equilibrium. □

B.2 Proof of Lemma 2

For the analysis we also consider

$$B(T, \mathbf{p}) = \{i \mid \exists (i, j, k) \in K(T, \mathbf{p})\},$$
$$B_a(T, \mathbf{p}) = \{i \mid \exists (i, j, k) \in K_a(T, \mathbf{p})\},$$

as the sets of buyers corresponding to $K(T, \mathbf{p})$ and $K_a(T, \mathbf{p})$, where $B_a(T, \mathbf{p}) \subseteq B(T, \mathbf{p})$.

We first focus on S_{r_1} with $r_1 = \max_j p'_j/p_j$, i.e., the set of goods with largest factor of price increase from $\mathbf{p}$ to $\mathbf{p}'$. For any $i \in B(S_{r_1}, \mathbf{p}')$, there is some $(i, j, k) \in K(S_{r_1}, \mathbf{p}')$ such that $u_{ijk}/p'_j \geq u_{ij'k'}/p'_{j'}$ for all $\mathbf{p}'$-active (i, j', k') with $j' \notin S_{r_1}$. Since $u_{ijk}/p_j = r_1 u_{ijk}/p'_j$ and $r_1 u_{ij'k'}/p'_{j'} > u_{ij'k'}/p_{j'}$ we conclude $K(S_{r_1}, \mathbf{p}') \subseteq K(S_{r_1}, \mathbf{p})$.

Next we analyze the total money spent on segments with $j \in S_{r_1}$ by buyers in $B(S_{r_1}, \mathbf{p}')$, with respect to equilibria E and E'. Since the prices p'_j of goods $j \in S_{r_1}$ decrease by the largest factor, the spending on these goods in E can only increase. In fact, we have that

$$\sum_{(i,j,k) \in K(S_{r_1}, \mathbf{p})} f_{ijk} \geq \sum_{(i,j,k) \in K(S_{r_1}, \mathbf{p})} f'_{ijk} \ , \quad \text{for every buyer } i \in B. \tag{5}$$

This implies

$$\begin{aligned} \sum_{j \in S_{r_1}} p_j^a &= \sum_{i \in B(S_{r_1}, \mathbf{p})} \sum_{(i,j,k) \in K(S_{r_1}, \mathbf{p})} f_{ijk} \\ &\geq \sum_{i \in B(S_{r_1}, \mathbf{p}')} \sum_{(i,j,k) \in K(S_{r_1}, \mathbf{p}')} f'_{ijk} = \sum_{j \in S_{r_1}} {p'_j}^a \ . \end{aligned} \tag{6}$$

However, since $p'_j > p_j$ for every $j \in S_{r_1}$, this can only be fulfilled when the inequalities in (5) and (6) are equalities. In particular, all goods in S_{r_1} must exactly reach their earning limit in both E and E' (as already observed in Proposition 1 part 2). Moreover, in E, no $i \in B(S_{r_1}, \mathbf{p}) \setminus B(S_{r_1}, \mathbf{p}')$ can ever receive allocation from goods in S_{r_1}. Hence, $B_a(S_{r_1}, \mathbf{p}) = B_a(S_{r_1}, \mathbf{p}')$.

In both E and E' each buyer $i \in B_a(S_{r_1}, \mathbf{p})$ spends the same amount of money on S_{r_1}, which we denote by $m_i(S_{r_1})$. Every buyer spends on segments in non-increasing order of u_{ijk}/p_j. This implies that a segment is $\mathbf{p}$-MBB iff it is

$\mathbf{p}'$-MBB. The possible allocations are the solution of a transportation problem, where each good $j \in S_{r_1}$ receives d_j flow, each buyer $i \in B_a(S_{r_1}, \mathbf{p})$ emits $m_i(S_{r_1})$ flow, routed over the same set of MBB edges in non-increasing order of bang-per-buck ratio. Every such allocation is a possible spending in both equilibria. This implies $K_a(S_{r_1}, \mathbf{p}) = K_a(S_{r_1}, \mathbf{p}')$. Note that $K_a(S_{r_1}, \mathbf{p}') \subset K(S_{r_1}, \mathbf{p}')$ when there are two $\mathbf{p}'$-active segments $(i, j, k), (i, j', k') \in K(S_{r_1}, \mathbf{p}')$ with $f_{ijk} = 0$ and $f_{ij'k'} > 0$.

In this sense, the spending and the way goods are allocated in $\mathbf{p}$ remains a feasible assignment on $\mathbf{p}'$-MBB segments. As such, we can drop the goods from S_1 from consideration. Then, we can apply the analysis in the same way for $r_2 = \max_{j \notin S_{r_1}} p'_j / p_j$ and S_{r_2}. Iterative application shows the properties for all S_r with $r > 1$; that is, $K_a(S_r, \mathbf{p}) = K_a(S_r, \mathbf{p}')$ and $K_a(S_r, \mathbf{p}) \subseteq K(S_r, \mathbf{p}') \subseteq K(S_r, \mathbf{p})$. Reversing the role of $E = (\mathbf{f}, \mathbf{p})$ and $E' = (\mathbf{f}', \mathbf{p}')$ we obtain the same claims for sets S_r with $r < 1$. That is, $K_a(S_r, \mathbf{p}') = K_a(S_r, \mathbf{p})$, $K_a(S_r, \mathbf{p}) \subseteq K(S_r, \mathbf{p}) \subseteq K(S_r, \mathbf{p}')$. Finally, since all segments $K_a(S_r, \mathbf{p}) = K_a(S_r, \mathbf{p}')$, for every $r \neq 1$, this must also hold for $r = 1$. This proves parts 1 and 2. Part 3 is a consequence of part 2 – since $K_a(S_r, \mathbf{p}) = K_a(S_r, \mathbf{p}')$, every $\mathbf{p}$-MBB segment with $f_{ijk} > 0$ is $\mathbf{p}'$-MBB and vice versa. This proves part 3 and concludes the proof. □

B.3 Proof of Theorem 2

We first observe the following fact about transportation problems on forests.

Lemma 9. *The solution for a transportation problem on a forest is unique.*

Proof. Let $e = (x, y)$ be any edge of the forest. Removal of e splits the tree containing e into two sets X and Y with $x \in X$ and $y \in Y$. The flow across e in the direction from x to y is $\sum_{u \in X} b(u) = -\sum_{v \in Y} b(y)$. Note that $\sum_{w \in X \cup Y} b(w) = 0$.

Alternatively, we may consider any edge (x, y) incident to a leaf x in the forest. Then the flow across the edge (x, y) is equal to $b(x)$. We add $b(x)$ to $b(y)$, remove x, and iterate. □

Consider the equilibrium $E = (\mathbf{f}, \mathbf{p})$ with smallest prices. Suppose there is another equilibrium $E' = (\mathbf{f}', \mathbf{p}')$ with prices $\mathbf{p}' \geq \mathbf{p}$. By Lemma 9, there are unique money flows in $\mathbf{f}$ and $\mathbf{f}'$ in E and E', respectively. Every good $j \in S_1$ with $p_j = p'_j$ has inflow p^a_j in both equilibria. Every good with $p'_j > p_j$ has inflow d_j in both equilibria due to Proposition 1 part 2. Every MBB segment (i, j, k) with $f_{ijk} > 0$ remains MBB under $\mathbf{p}'$ due to Lemma 2 part 3. Thus, $\mathbf{f}$ remains a feasible flow for E'. Since by Lemma 9 money flows are unique, we have $\mathbf{f} = \mathbf{f}'$. □

B.4 Proof of Lemma 3

The surplus of a good can only decrease if its price decreases or if additional money flow is pushed into it – in particular, observe that the adjustment of the flow to Δ-discrete capacities only increases the surplus of each good. If additional money flow is pushed into a good, its surplus before the push is at least Δ. Hence, it is non-negative after the push. Price decreases stop once there is a good with a non-positive surplus, so a non-negative surplus cannot become negative. □

B.5 Proof of Lemma 4

After initialization, all buyers have surplus $\ell\Delta \leq s(i) \leq (\ell+1)\Delta$ and all goods have surplus $s(j) \leq n(\ell+1)\Delta$. In the beginning of the outer while-loop, we reduce Δ to half and adjust the flow to Δ-discrete capacities. Due to reduction of Δ, all buyers have surplus $2\ell\Delta \leq s(i) \leq 2(\ell+1)\Delta$ and all goods have surplus $s(j) \leq 2n(\ell+1)\Delta$. Due to adjustment of the flow to Δ-discrete capacities, $s(i)$ decreases by at most $\ell\Delta$, and $s(j)$ increases by at most $\ell\Delta$, for every $i \in B$, $j \in G$. This results in $\ell\Delta \leq s(i) \leq 2(\ell+1)\Delta$ and $s(j) \leq 2n(\ell+1)\Delta + \ell\Delta$. In the following loop, we reduce the surplus of all buyers to $\ell\Delta \leq s(i) \leq (\ell+1)\Delta$, which takes at most $n(\ell+1)$ iterations. This implies that every buyer surplus satisfies the conditions of a Δ-feasible solution. Every buyer surplus stays unchanged in the Δ-phase.

In the subsequent Δ-phase, we reduce the surplus of every good to at most Δ. All prices are non-increasing, hence without flow adjustment all surpluses of goods are non-increasing. In a flow adjustment along path P, we keep every surplus of intermediate goods the same. We reduce the surplus of good j' and increase the surplus of good j by Δ. Since good j has non-positive surplus, this never increases the surplus beyond Δ. Since good j' has surplus more than Δ, this never makes the surplus of good j' negative.

Hence, in the first Δ-phase there can be at most n iterations that terminate with Event 1, and at most $2n^2(\ell+1)+n\ell$ that terminate with Event 2. Furthermore, since prices are decreasing, Event 3 happens at most n times overall. Moreover, since the residual network $\mathcal{G}_r$ expands at most n times by including a new buyer, Event 4 happens at most n times in each iteration. Overall, the first Δ-phase takes time at most $n(\ell+1) + n(n + 2n^2(\ell+1) + n\ell) + n = O(n^3\ell)$.

Note that at the end of the Δ-phase, we have a Δ-feasible solution. The conditions for the surplus of all buyers hold, since they were unchanged during the Δ-phase. By Lemma 3, we have negative surplus only for goods whose price has not been touched in the process. By termination of the Δ-phase, it follows that every good surplus satisfies the conditions of Δ-feasible solution.

Hence, in every subsequent run of the outer while-loop, we start with $s(j) \leq \Delta$ for all goods. After adjustment of Δ and the flow to Δ-discrete capacities, we have $s(j) \leq (\ell+2)\Delta$ for every $j \in G$ and $\ell\Delta \leq s(i) \leq 2(\ell+1)\Delta$ for every $i \in B$. The next for-loop then guarantees $s(i) \leq (\ell+1)\Delta$ for all buyers. By repeating the arguments above, the following Δ-phase takes time $O(n^2\ell)$. □

B.6 Proof of Lemma 5

Note that the flow values are always Δ-integral, hence they are rational numbers with desired size. Also, the starting prices are rational numbers of desired size. At the end of each iteration, one of the four events occurs. In all cases, we show that prices remain polynomially bounded if they are so at the beginning of the iteration. This will complete the proof.

In case of Event 3, a capped good j becomes uncapped, so $p_j = d_j$ and the ratio of any other price in the active component and p_j can be written as the ratio of product of at most n utility values. Hence, they are polynomially bounded. The other prices are not touched, so they remain same.

In case of Event 4, a new active segment arises, and therefore we can again write any price in the active component in terms of a price variable which has not been touched. All prices are polynomially bounded.

Event 1 can happen only if $p^a_k = p_k$. In that case, p_k is Δ-integral and all other prices in the active component can be expressed in terms of p_k using the MBB relation. Hence, all prices are of desired size.

In case of Event 2, if $s(j) < 0$, then this implies that p_j has not been decreased since the beginning, so all prices are again fine. For the other case, $s(j) = 0$ and that implies p_j is Δ-integral. Therefore, all prices are polynomially bounded. □

B.7 Proof of Lemma 6

Consider the connected components of bipartite graph $(B \cup G, E)$, where $E = \{(i, j) \in B \times G \mid f_{ij} > 0\}$. We show the claim for each connected component C separately. If there is a good j with negative surplus, then $p_j = p^0_j$. This implies that $p_h \geq d_h$ and $p^a_h = d_h$ for all goods $h \in C \cap G$. Hence the claim follows from (MC). If all goods have non-negative surplus,

$$\begin{aligned}\sum_{j\in N(\hat{B})} p^a_j - \sum_{i\in\hat{B}} m_i &= \sum_{j\in N(\hat{B})}\left(p^a_j - \sum_{i\in\hat{B}} f_{ij}\right) + \sum_{i\in\hat{B}}\left(\sum_{j\in N(\hat{B})} f_{ij} - m_i\right)\\ &\geq \sum_{j\in N(\hat{B})}\left(p^a_j - \sum_{i\in B} f_{ij}\right) + \sum_{i\in\hat{B}}\left(\sum_{j\in N(\hat{B})} f_{ij} - m_i\right)\\ &= \sum_{u\in\hat{B}\cup N(\hat{B})} s(u) \geq 0\ .\end{aligned}$$

□

B.8 Proof of Lemma 7

The first claim follows from Lemma 4. Thus, $s(j) \leq \Delta$ for every good j. By Lemma 6

$$0 \leq \sum_{j\in G} p^a_j - \sum_{i\in B} m_i = \sum_{u\in B\cup G} s(u) \leq n(\ell+1)\Delta - \sum_{u;\ s(u)<0} \|s(u)\|\ ,$$

and hence $s(j) \geq -n(\ell+1)\Delta$ for every good j. □

B.9 Proof of Theorem 3

At the beginning, $\Delta \leq U^{n+1}$ and Δ is reduced to $\Delta/2$ until $\Delta < 1/(2\ell(2nU)^{4n})$. Therefore, since $\ell \geq n$, the total number of Δ-phases is $O(n \log(\ell + nU))$. While the first phase takes time $O(n^3\ell)$, each subsequent phase takes time $O(n^2\ell)$. Further, PostProcessingS takes $O(n^4 \log(nU))$ time [11]. The total running time of Algorithm 1 is $O(n^3\ell \log(\ell + nU))$. □

B.10 Proof of Theorem 4

By Lemma 2 part 3, we know that if $f_{ijk} > 0$, then (i, j, k) is an MBB segment in every thrifty equilibrium. Let $E(\mathbf{f}) = \{(i, j, k) \mid f_{ijk} > 0\}$, and let G_c and G_u be the (unique) sets of capped and uncapped goods in thrifty equilibria, respectively. Note that a vector $\mathbf{q}$ of pointwise smallest prices implies a pointwise largest MBB ratio α_i for all buyers $i \in B$. Using $\lambda_i = 1/\alpha_i > 0$, Algorithm 2 optimizes the LP to find the minimal λ_i with prices $\mathbf{q}$ that preserve the MBB segments. The prices $\mathbf{q}$ then determine all active segments, and they determine the flow on all segments that are non-active and MBB (and, thus, closed). For the active ones, the feasible flows are exactly the solutions to a straightforward transportation problem. In particular, the original flow $\mathbf{f}$ stays an equilibrium flow, since all edges that carry flow in $\mathbf{f}$ stay MBB, the outflow of every buyer i is m_i, the inflow of every good j is p_j^a. Moreover, $\mathbf{f}$ saturates non-active MBB segments under prices $\mathbf{q}$, which is directly implied by the proof of Lemma 2, part 2. □

B.11 Proof of Theorem 5

It is easy to check that throughout the algorithm, $(\mathbf{f}, \mathbf{p})$ always remains a thrifty equilibrium. Assume by contradiction that at the end of the algorithm, $(\mathbf{f}, \mathbf{p})$ is not an equilibrium with largest prices. Let $E' = (\mathbf{f}', \mathbf{p}')$ be an equilibrium with largest prices, and define $S_1 = \{j \mid p'_j > p_j\}$. By Proposition 1 part 2, all goods in S_1 are capped goods. Moreover, by Lemma 2 part 3 every segment with $f_{ijk} > 0$ for $j \in S_1$ is also an MBB segment in E'. Because prices of goods in S_1 strictly decrease from $\mathbf{p}'$ to $\mathbf{p}$, every buyer i with active edges in S_1 in the active segment graph with prices $\mathbf{p}'$ will have active edges only to S_1 with prices $\mathbf{p}$. Therefore, set S is nonempty for the While loop, and the algorithm should not terminate. □

B.12 Proof of Proposition 2

First, we replace each item with $d_j \leq 2n$ by d_j auxiliary items with supply 1 as in the direct adjustment. Each of these gets an auxiliary good with earning limit 1 in the market. For each item with $d_j > 2n$, we introduce an uncapped good in the market. For every auxiliary good, we assume that every buyer i has utility $u_{ij} = v_{ij}$. For every uncapped good, we assume every buyer i has utility $u_{ij} = v_{ij}d_j$. Then we use our algorithm above to compute a thrifty equilibrium for this market in time $O((nm)^4 \log((nm) \cdot \max_{i,j} v_{ij} d_j))$. Let $(\mathbf{x}, \mathbf{p})$ be this equilibrium.

The subsequent rounding of the equilibrium allocation follows [8]. Consider the spending graph, i.e., the subgraph of the MBB graph where buyers spend their money. Because of non-degeneracy of the MBB graph [8, 16], the spending graph is a forest. To handle the uncapped goods, we first present an inefficient approach and then observe how to implement it implicitly in polynomial time.

Given an uncapped good j, let us expand the spending graph in the following way: Introduce d_j many copies, each with price $p'_j = p_j/d_j$. The valuation of buyer i for each copy is v_{ij}. Since good j is uncapped, we know $p_j \leq \sum_i m_i = n$. Moreover, since $d_j > 2n$, this implies $p'_j < 1/2$. Let $f_{ij} = x_{ij} p_j$ be the money that agent i spends on good j. The parent agent i_0 in the spending graph becomes direct parent of $\lceil f_{i_0,j}/p'_j \rceil$ many copies. If $f_{i_0,j}/p'_j$ is not integer, the parent pays p'_j to $\lfloor f_{i_0,j}/p'_j \rfloor$ many copies, and the rest to one additional copy. The first child i_1 of good j is assigned to contribute the missing money for this additional copy (until it is fully paid for) and becomes its child. Then, if i_1 still has remaining money, it contributes this money to purchase further copies, for which it becomes the parent. Also, it remains parent of any other goods $j' \neq j$ for which it is a parent in the spending graph. Naturally, if i_0 exactly pays an integer number of copies, i_1 becomes the root of a new tree component and purchases additional copies of good j in the same way.

More formally, i_1 becomes parent of

$$\max\left(0, \left\lceil \left(f_{i_1,j} - \left(p'_j \cdot \lceil f_{i_0,j}/p'_j \rceil - f_{i_0,j}\right)\right) / p'_j \right\rceil\right)$$

further copies of good j. We continue this expansion process, in which child agent i_k of good j becomes parent of

$$\max\left(0, \left\lceil \left(f_{i_k,j} - \left(p'_j \cdot \left\lceil \sum_{\ell=0}^{k-1} f_{i_\ell,j}/p'_j \right\rceil - \sum_{\ell=0}^{k-1} f_{i_\ell,j}\right)\right) / p'_j \right\rceil\right) \quad . \tag{7}$$

many copies of good j. Since prices and utilities are both scaled by d_j, it is easy to verify that this represents an equilibrium assignment for the market where we introduce d_j auxiliary goods for good j, each with earning limit 1.

Now, since $p'_j < 1/2$, the rounding procedure in [8] applied to this expanded spending graph will assign all copies of item j to the parent agent of the corresponding good and remove them from the graph. Thus, the rounding procedure simply removes good j from the spending graph and assigns the number of copies given by (7) to parent buyer i_0 and children i_ℓ, for $\ell = 1, 2, \ldots$. Obviously, this can be done directly for each uncapped good j in $O(n)$ time without explicit expansion of the spending graph. Consequently, our adjusted algorithm achieves a running time of $O(n^4 \log nU)$ because our algorithm takes $O(n^4 \log nU)$ time and the rounding procedure takes $O(n^4)$ time. □