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Stable Matching with Network Externalities

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Abstract We study the stable roommates problem in networks where players are embedded in a social context and may incorporate positive externalities into their decisions. Each player is a node in a social network and strives to form a good match with a neighboring player. We consider the existence, computation, and inefficiency of stable matchings from which no pair of players wants to deviate. We characterize prices of anarchy and stability, which capture the ratio of the total profit in the optimum matching over the total profit of the worst and best stable matching, respectively. When the benefit from a match (which we model by associating a reward with each edge) is the same for both players, we show that externalities can significantly improve the price of stability, while the price of anarchy remains unaffected. Furthermore, a good stable matching achieving the bound on the price of stability can be obtained in polynomial time. We extend these results to more general matching rewards, when players matched to each other may receive different benefits from the match. For this more general case, we show that network externalities (i.e., "caring about your friends") can make an even larger difference and greatly reduce the price of anarchy. We show a variety of existence results and present upper and lower bounds on the prices of anarchy and stability for various structures of matching benefits. All our results on stable matchings immediately extend to the more general case of fractional stable matchings.

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1 Introduction

Stable matching problems capture the essence of many important assignment and allocation tasks in economics and computer science. The central approach to analyzing such scenarios is two-sided matching, which has been studied intensively since the 1970s in both the algorithms and economics literature. Stable matchings are a special case of the much more general stable roommates problem which forms the basis of this work. An important variant of stable matching is matching with cardinal utilities, when each match can be given numerical values expressing the *quality* or *reward* that the match yields for each of the incident players [6]. Cardinal utilities specify the quality of each match instead of just a preference ordering, and they allow the comparison of different matchings using measures such as social welfare. A particularly appealing special case of cardinal utilities is known as correlated stable matching, where both players who are matched together obtain the same reward. Apart from the wide-spread applications of correlated stable matching in, e.g., market sharing [24], job markets [9], social networks [26], and distributed computer networks [42], this model also has favorable theoretical properties. It guarantees existence of a stable matching and convergence of dynamics even in the non-bipartite case, where every pair of players is allowed to match [3, 4, 42]. However, it should also be mentioned that in many scenarios agents may not be able to express their preferences in terms of cardinal utilities or may only be able to do so approximately.

When matching individuals in a social environment, it is often unreasonable to assume that each player cares only about their own match quality. Instead, players incorporate the well-being of their friends/neighbors as well, or that of friends-of-friends. Players may even be altruistic to some degree, and consider the welfare of all players in the network. Such network externalities are commonly observed in practice and have been documented in laboratory experiments [39, 21]. In addition, in economics there exist recent approaches towards modeling and analyzing *other-regarding preferences* [22]. Given that other-regarding preferences are widely observed in practice, it is an important fundamental question to model and characterize their influence in classic game-theoretical environments. Very recently, the impact of social influence on congestion and potential games has been characterized prominently in [11, 27, 16, 18, 29, 17, 28].

In this paper, we consider a natural approach to incorporate externalities from social networks into partner selection and matching scenarios. In particular, we study how social context influences stability and efficiency in matching games. Our approach uses dyadic influence values tied to the hop distance in the graph. In this way, every player may consider the well-being of every other player to some degree, with the degree of this regardfulness possibly decaying with the hop distance. The perceived utility of a player is then composed of a weighted average of match utilities. Players who only care about their neighbors, as well as fully altruistic players, are special cases of this model.

Moreover, for matching in social environments, the standard model of correlated stable matching may be too constraining compared to general cardinal utilities, because matched players receive exactly the same reward. Such an *equal sharing* property is intuitive and bears a simple beauty, but there are a variety of other reward sharing methods that can be more natural in different contexts. For instance, consider pairs of students working together on class projects. Should both students in a pair receive the same grade/reward, or should they be given a grade based on how much effort they put into the project? In another example, when joint projects result in a publication, in mathematics and theoretical computer science it is common practice to list authors alphabetically, but in other disciplines the author sequence is carefully designed to ensure a proper allocation of credit to the different participants of a joint paper. Here the credit is often supposed to be allocated in terms of input, i.e., the first author should be the one that has contributed most to the project. Such input-based or proportional sharing is then sometimes overruled with sharing based on intrinsic or acquired social status, e.g., when a distinguished expert in a field is easily recognized and subconsciously credited most with authorship of an article. In this paper, we are interested in how such unequal reward sharing rules affect stable matching scenarios. In particular, we consider a large class of local reward sharing rules and characterize the impact of unequal sharing on existence and inefficiency of stable matchings, both in cases when players are embedded in a social context and when they are not.

1.1 Stable Matching Within a Social Context

Correlated stable matching games is a prominent subclass of general cardinal stable matching games. In this game, we are given a (non-bipartite) graph $G = (V, E)$ with edge weights $r_e > 0$ for all $e \in E$. Every node is a player and strives to build (at most) one incident edge from E. A matching $M \subseteq E$ is a non-overlapping set of edges. In a matching M, if node u is unmatched, the reward of node u is 0. Otherwise, if u is matched to node v, the reward of node u is defined to be exactly r_e where e denotes the edge (u, v). This can be interpreted as both u and v getting an identical reward from being matched together. We will also consider unequal reward sharing, where for each $e = \{u, v\} \in E$ in addition to r_e we have two rewards $r_e^u, r_e^v \geq 0$ with $r_e^u + r_e^v = r_e$. Therefore, the preference ordering of each node over its possible matches is implied by the rewards that this node obtains from different edges. A pair of nodes $\{u, v\}$ is called a *blocking pair* in matching M if u and v are not matched to each other in M, but can both strictly increase their rewards by being matched to each other instead. A matching with no blocking pairs is called a *stable matching.*

While the matching model above has been well-studied, in this paper we are interested in stable matchings with externalities that arise in the presence of social context. Denote the reward obtained by a node v in a matching M

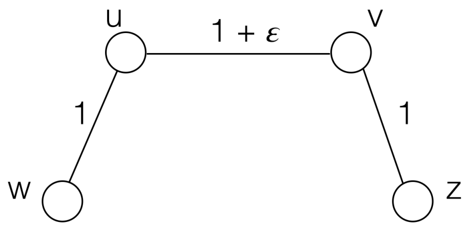

Fig. 1 The path of four nodes as described in Example 1

as R_v. We now consider the case when node u not only cares about its own reward, but also about the rewards of its friends. Specifically, for node $v \in V$ we denote by $N_d(v)$ the set of nodes that have shortest distance of exactly d to v in G. Then the *perceived utility* or *friendship utility* of node v in matching M is defined as

$$U_v(M) = R_v(M) + \sum_{d=1}^{diam(G)} \alpha_d \sum_{u \in N_d(v)} R_u(M), \quad (1)$$

where $1 \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq 0$ (we use $\boldsymbol{\alpha}$ to denote the vector of α_i values). In other words, for a node u that is distance d away from v, the utility of v increases by an α_d factor of the reward received by u. Thus, if $\alpha_d = 0$ for all $d \geq 2$, this means that nodes only care about their neighbors, while if all $\alpha_d > 0$, this means that nodes are altruistic and care about the rewards of everyone in the graph. The perceived utility is the quantity that the nodes are trying to maximize, and thus, in the presence of externalities, a blocking pair is a pair of nodes such that each node can increase its *perceived utility* by matching to each other. Using this definition of blocking pair, a stable matching is again a matching without such a blocking pair.

Example 1. Consider a simple path of four nodes (see Fig. 1), with edges $e_1 = \{w, u\}, e = \{u, v\}, e_2 = \{v, z\}$. This structure will play a prominent role throughout the paper. First assume that $r_{e_1} = r_{e_2} = 1$ and $r_e = 1 + \varepsilon$, for $\varepsilon > 0$. We consider equal sharing, that is, both incident players receive the same reward from an edge. For small values of ε, the matching with maximum total reward is $M^* = \{e_1, e_2\}$. For $\boldsymbol{\alpha} = 0$, however, the unique stable matching is $M_s = \{e\}$, as both u and v can strictly improve their rewards for every matching M with $e \notin M$.

Now assume that $\varepsilon < 0.33$ and we have friendship utilities with $\alpha_1 = 0.5$ and $\alpha_2 = \alpha_3 = 0$. It is easy to observe that M_s remains a stable matching. Consider one of the possible pairs $\{u, w\}$ and $\{v, z\}$. In each one, the perceived utility of u or v would go down from $1.5 \cdot (1 + \varepsilon)$ to 1.5, respectively. Hence, none of them constitutes a blocking pair and M_s is stable. If, however, both of these edges are constructed, then M^* is also a stable matching. The only possible blocking pair e destroys both $\{u,w\}$ and $\{v,z\}$. Then the perceived utility deteriorates from 2 to $1.5 \cdot (1+\varepsilon) < 2$ for both u and v. Hence, e is not a blocking pair, and M^* is stable. ■

We also study more general stable matching models with nodes u and v receiving different rewards r^u_{uv} and r^v_{uv} from an edge $\{u,v\} \in M$. Under these conditions, a stable matching is not guaranteed to exist. Instead, we resort to *fractional stable matchings* defined as follows. In a fractional matching M there is a real number $x_e \in [0,1]$ for each edge e. It represents the degree to which edge e is "in the matching" and can be thought of as the strength of the match between the nodes incident to e. In addition, for every node u there is a budget constraint $\sum_{e \in E, e \ni u} x_e \leq 1$. Fractional matching is especially well-motivated in a social context, since it captures the idea of relationships of varying strengths, e.g., time spent together by friends or the extent of collaboration between two people, etc. The budget constraint models the fact that a single person cannot be involved in an unlimited amount of strong relationships. With fractional link strengths, the reward of a node u for an edge e becomes $x_e \cdot r^u_e$, and the total reward of node u becomes $\sum_{e \in E, e \ni u} x_e r^u_e$.

A fractional stable matching is a fractional matching without blocking pairs. Analogous to a blocking pair for an integral matching, a blocking pair for a fractional matching is an edge $\{u,v\}$ such that by increasing the strength of edge $\{u,v\}$ (and possibly decreasing the strengths of some other edges $\{u,w\}$ and $\{v,z\}$ to keep the budget constraints), both u and v strictly improve their utilities. For fractional matching, the extension to friendships, social context, and perceived utility is straightforward. Throughout the paper, the term *stable matching* refers to an integral stable matching. We will explicitly mention when fractional stable matchings are studied.

Example 1 (contd.)**.** For our path example with equal sharing above without friendship and $\boldsymbol{\alpha} = 0$, M_s is the unique stable matching. It is also the unique fractional stable matching, as every fractional matching M that has $x_e < 1$ allows u and v to jointly increase x_e to 1 (and thereby possibly decreasing x_{e_1} and x_{e_2}). In every case, this represents an increase of the reward of both u and v.

For the case of friendship utilities with $\alpha_1 = 0.5$ and $\varepsilon < 0.33$, every linear combination between the stable matchings M_s and M^* constitutes a fractional stable matching, and it is easy to see that these are the only fractional stable matchings. For instance, $x_{e_1} = 0.75$, $x_e = 0.25$ and $x_{e_2} = 0.75$ is one such fractional stable matching. A small increase of δ on either x_{e_1} or x_{e_2} leads to a decrease on x_e by the same amount. The changes in perceived utility are the same as observed above (times δ), so this represents a deterioration of the reward of u or v, respectively. An increase on e is not profitable for neither u nor v for the same reasons. ■

Centralized Optimum and the Price of Anarchy. We study the social welfare of stable matchings and compare them to an optimal centralized matching.

The social welfare $v(M)$ of a matching M is defined as the sum of rewards obtained by all the nodes in the matching M, i.e.,

$$v(M) = \sum_u R_u(M) \ .$$

A *social optimum* or a *socially optimal matching*, which we will denote by M^*, is a matching with the maximum social welfare, i.e.,

$$M^* = \underset{M \text{ is a matching}}{\arg\max} \ v(M) \ .$$

While this is equivalent to maximizing the sum of player utilities when $\boldsymbol{\alpha} = 0$, this is no longer true for perceived utilities with externalities $\boldsymbol{\alpha} \neq 0$. Nevertheless, similar to, e.g. [43, 17], we believe this is a well-motivated and important measure of solution quality, as it captures the overall performance of the system, while ignoring the perceived "good-will" effects of friendship and altruism. For example, when considering projects done in pairs, the reward of an edge can represent actual productivity, while the perceived utility may not. Also note that a socially optimal matching need not be a stable matching.

To compare stable solutions with a social optimum, we will often consider the price of anarchy (PoA) and the price of stability (PoS), defined over the space $\mathcal{I}$ of all instances as follows:

$$\text{PoA} \ = \ \max_{I \in \mathcal{I}} \ \max_{\substack{M, M^* \text{are stable and} \\ \text{socially optimal resp. in } I}} \frac{v(M^*)}{v(M)} \ .$$

$$\text{PoS} \ = \ \max_{I \in \mathcal{I}} \ \min_{\substack{M, M^* \text{are stable and} \\ \text{socially optimal resp. in } I}} \frac{v(M^*)}{v(M)} \ .$$

Intuitively speaking, PoA quantifies the maximum possible gap between a socially optimal matching and a stable matching in any instance. PoS quantifies the gap to a socially optimal matching within some stable matching is guaranteed to exist in any instance.

Example 1 (contd.)**.** Reconsider Example 1. If $\boldsymbol{\alpha} = 0$, the total reward of the optimum matching M^* is 4, while the reward of the unique (i.e., best and worst) integral and fractional stable matching M_s is $2(1+\varepsilon)$. For $\varepsilon \to 0$, the example constitutes a lower bound on prices of anarchy and stability of 2 (which is also the correct upper bound, see, e.g., [8]). In this example, if we start with $\boldsymbol{\alpha} = 0$ and begin increasing α_1 until it reaches $\alpha_1 = 1$, we see the following effects. (1) The set of stable matchings only expands, i.e., matchings that are stable at small α_1 values remain stable as we increase α_1. (2) The price of anarchy remains the same for all values of α_1. (3) The price of stability improves as α_1 increases: more specifically in this example it remains 2 until α_1 becomes large enough, and then becomes equal to 1. Below, we will show that these properties hold much more generally, and not just for this simple example. ■

1.2 Our Results

Equal Sharing In Section 2, we consider stable matching with friendship utilities and equal reward sharing. Our first insight here is a monotonicity result about the set of stable matchings – given any game, when we increase an entry of $\boldsymbol{\alpha}$, the set of stable matchings only expands. This implies that a stable matching exists, the price of anarchy (ratio of the maximum-weight matching with the worst stable matching) can only increase, and the price of stability (ratio with the best stable matching) can only decrease. In fact, the price of anarchy remains at most 2, the same as in the case without friendship. The price of stability, on the other hand, improves in the presence of friendship, as we can show a tight bound of $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$. Moreover, we present a dynamic process that converges to a stable matching of at least this quality in polynomial time if initiated from a social optimum matching. Our results imply that with network externalities, the price of stability can greatly improve: e.g., if $\alpha_1 = \alpha_2 = \frac{1}{2}$, then the price of stability is at most $\frac{6}{5}$, and a solution of this quality can be obtained efficiently.

The price of stability relies only on α_1 and α_2, because the existence of a blocking pair $\{u, v\}$ depends only on the α values of nodes in distance of at most 2 from u and v. This is because by deviating, u and v cannot affect the matches of any nodes which are farther than distance 2 away from them. To construct a good stable matching we have to ensure resilience to such deviations, and as their effect is "local" in this sense, this locality is reflected also in the bound.

General Reward Sharing In Section 3 we study more general reward sharing schemes. When two nodes matched together may receive different rewards, an integral stable matching may not exist. Thus, we focus on *fractional* stable matchings which we show to always exist, even with friendship utilities. We show that for arbitrary reward sharing, prices of anarchy and stability depend on the level of inequality among reward shares. Specifically, if R is the maximum ratio over all edges $\{u, v\} \in E$ of the reward shares of node u and v, then the price of anarchy is at most $\frac{(1+R)(1+\alpha_1)}{1+\alpha_1 R}$. Thus, compared to the equal reward sharing case, if sharing is extremely unfair (R is unbounded), then externalities become even more influential: changing α_1 from 0 to $\frac{1}{2}$ reduces the price of anarchy from unbounded to at most 3. In addition, for several particularly natural local reward sharing rules, we show that an integral stable matching exists, give improved price of anarchy guarantees, and show tight lower bounds.

General Additive-Separable Externalities Some of our results continue to hold when the values of α are not tied to hop distance. In particular, suppose for each unordered pair $u, v \in V$ there are values $\alpha_{uv}^u \geq 0$ and $\alpha_{vu}^v \geq 0$, possibly independent of their distance, and perceived utility of a node v in a matching M is given by $U_v(M) = R_v(M) + \sum_{u \neq v} \alpha_{uv}^v R_u(M)$.

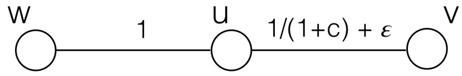

Fig. 2 An instance for which bad matchings become stable for friendship utilities with general additive-separable externalities (See Example 2)

For equal sharing and $\alpha_{uv} = \alpha_{uv}^u = \alpha_{uv}^v$, a stable matching under friendship utilities still exists. However, the set of stable matchings can change completely when increasing $\boldsymbol{\alpha}$ from $\mathbf{0}$. Both prices of anarchy and stability can strictly increase in this case.

Example 2. As shown in Fig. 2, consider a path of length 2 with edges $e_1 = \{w, u\}$ and $e = \{u, v\}$. Suppose $r_{e_1} = 1$ and $r_e = \frac{1}{1+c} + \varepsilon$, for $c > 0$ and $\varepsilon \in (0, c/(1+c))$. In this case, the optimum is edge e_1, and it is also the unique stable matching when $\boldsymbol{\alpha} = \mathbf{0}$. Hence, both prices of anarchy and stability are 1 in this case. When we increase α_{uv} from 0 to some value larger than c, then edge e becomes the unique stable matching, and for $\varepsilon \to 0$ the prices of anarchy and stability become $1 + c$. ■

For $\alpha_{uv} = c \in [0, 1]$, the above example provides both a price of anarchy and stability lower bound of up to 2. In fact, there are games where the price of anarchy can increase even beyond 2.

Example 3. Consider the same path structure as in Example 1 with edges $e_1 = \{w, u\}, e = \{u, v\}, e_2 = \{v, z\}$. Here we assume that rewards are $r_{e_1} = r_{e_2} = 1$ and $r_e = 0.5$. Also, assume $\alpha_{uv} = 1$ and all other values of the vector α to be 0. It is straightforward to see that edge e is a stable matching here, and the price of anarchy becomes 4. ■

For general reward sharing, the existence of fractional stable matchings even continues to hold when for each unordered pair $u, v \in V$ there are possibly different non-negative $\alpha_{uv}^u \neq \alpha_{uv}^v$. However, our bounds on the price of anarchy cannot be extended to this case. A deeper study of the price of anarchy with general externalities is an interesting avenue for future work.

1.3 Related Work

Stable matching problems have been studied intensively over the last few decades. On the algorithmic side, existence, efficient algorithms, and improvement dynamics for two-sided stable matchings have been of interest (for references, see standard textbooks [25, 46, 40]). In this paper, we address the more general stable roommates problem, in which every player can be matched to every other player. For general preference lists, there have been numerous works characterizing and algorithmically deciding existence of stable matchings [34, 48, 19, 46]. In contrast, fractional stable matchings are always guaranteed to exist and exhibit interesting polyhedral properties [2, 1, 48]. For the correlated stable roommates problem, existence of (integral) stable matchings is guaranteed by a potential function argument [3, 42], and convergence time of random improvement dynamics is polynomial [4]. In [7], bounds on prices of anarchy and stability for *approximate* correlated stable matchings were provided. Similar studies in a setting with geometric distances were conducted in [10]. In contrast, we study friendship, altruism, and unequal reward sharing in stable roommates problems with cardinal utilities.

Another line of research closely connected to some of our results involves game-theoretic models for contribution. In [8] we consider a contribution game tied closely to matching problems. Here players have a budget of effort and contribute parts of this effort towards specific projects and relationships. For more related work on the contribution game, see [8]. All previous results for this model concern equal sharing and do not address the impact of the player's social context. As we mention in the conclusion, many of our results for friendship utilities can also be extended to such contribution games.

Analytical aspects of reward sharing have been a central theme in game theory since its beginning, especially in cooperative games [44]. Recently, there have been prominent algorithmic results also for network bargaining [37, 35] and credit allocation problems [36]. Another recent line of work [49, 50] treats extensions of cooperative games, where players invest into different coalitional projects. The main focus of this work is global design of reward sharing schemes to guarantee cooperative stability criteria. Our focus here is closer to, e.g., recent work on profit sharing games [12, 41]. We are interested in existence, computational complexity, and inefficiency of stable states under different reward sharing rules, with an aim to examine the impact of social context on stable matchings. Closely related to our approach is a recent study of designing and allocating reward shares to reduce the price of anarchy and stability in stable matching scenarios without externalities [31].

Our notion of a player's social context is based on numerical influence parameters that determine the impact of player rewards on the (perceived) utilities of other players. A recently popular model of altruism is inspired by Ledyard [38] and has generated much interest in algorithmic game theory [18, 17, 29]. In this model, each player optimizes a perceived utility that is a weighted linear combination of his own utility and the utilitarian welfare function. Similarly, for surplus collaboration [11] perceived utility of a player consists of the sum of players utilities in his neighborhood within a social network. Our model is similar to [16, 28, 20, 45] and smoothly interpolates between these global and local approaches. The idea of utility being influenced by friends and peers has also been explored in the contexts of many-to-one two-sided matchings in housing allocation [14] and social choice [47]. Among the several key differences between our work and [14], our work is not restricted to two-sided matchings, and our definition of stability does not include edges as the additional players (unlike their case, where houses also have preferences

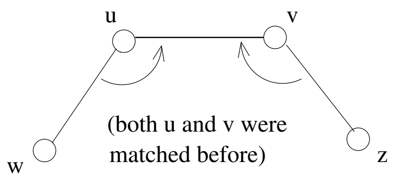

Fig. 3 biswivel deviation

over sets of occupants). Moreover, in our paper social welfare sums the actual utilities and not the perceived utilities. Another related work to assignment problems and social context is [13], a different approach where they attempt to infer and quantify different network externalities based on the dynamics of the assignment process. In [15] the authors describe how "attitudes" of agents can affect the agent deviations and the stability of matchings. However, their focus is the computational complexity of the dynamics and, unlike in our work, the perceived utility of an agent is not affected by the match quality of other agents.

Very recently, our model has been analyzed in terms of convergence of dynamics. The case of equal sharing and $\alpha_{uv} = \alpha^u_{uv} = \alpha^v_{uv}$ can be captured within a framework of consistent matching games, which encompass also other classes of stable matching with visibility and externality constraints [26, 32, 30]. In [33] it is shown that, from every starting matching in such a game, there is a sequence of polynomially many blocking-pair resolutions to a stable state. Moreover, every state reachable via a sequence of blocking-pair resolutions can also be reached within a polynomial number of steps. In contrast to our paper, these works do not address prices of anarchy and stability and the computation of good matchings.

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2 Stable Matching with Equal Reward Sharing

We begin by considering correlated stable matching in the presence of friendship utilities. In this section, the reward received by both nodes of an edge in a matching is the same, i.e., we use equal reward sharing, where every edge e has an inherent value r_e and both endpoints receive this value if edge e is in the matching. We consider more general reward sharing schemes in Section 3. Recall that the friendship utility of a node v increases by $\alpha_d R_u$ for every node u, where d is the shortest distance between v and u. We abuse notation slightly, and let α_{uv} denote α_d, so if u and v are neighbors, then $\alpha_{uv} = \alpha_1$.

Given a matching M, let us classify the following types of improving deviations that a blocking pair can undergo.

Definition 1. *We call an improving deviation a* **biswivel** *whenever two neighbors u and v switch to match to each other, such that both u and v were matched to some other nodes before the deviation in M.*

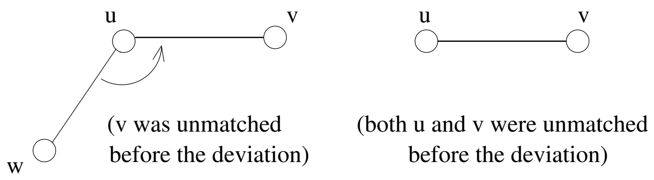

Fig. 4 swivel deviation

See Figure 3 for explanation. For such a biswivel to exist in a matching, the following necessary and sufficient conditions must hold.

$$(1+\alpha_1)r_{uv} > (1+\alpha_1)r_{uw} + (\alpha_1+\alpha_{uz})\, r_{vz} \tag{2}$$

$$(1+\alpha_1)r_{uv} > (1+\alpha_1)r_{vz} + (\alpha_1+\alpha_{vw})\, r_{uw} \tag{3}$$

Intuitively, the left side of Inequality (2) quantifies the utility gained by u because of getting matched to v and the right side quantifies the utility lost by u because of u and v breaking their present matchings with w and z respectively. Hence, Inequality (2) implies that u gains more utility by getting matched with v than it loses because of u and v breaking their matchings with v and z. Inequality (3) can similarly be explained in the context of node v.

Definition 2. *We call an improving deviation a* **swivel** *whenever two neighbors get matched such that at least one node among the two neighbors was not matched before the deviation.*

See Figure 4 for explanation. For such a swivel to occur, the following set of conditions must hold.

$$(1+\alpha_1)r_{uv} > (1+\alpha_1)r_{uw} \tag{4}$$

$$(1+\alpha_1)r_{uv} > (\alpha_1+\alpha_{vw})r_{uw} \tag{5}$$

Inequality (4) says that u gains more utility by getting matched with v than it loses by breaking its matching with w. Inequality (5) says that v gains more utility by getting matched with u than the utility it loses because of u breaking its matching with w. As $\alpha_1+\alpha_{vw} \leq 1+\alpha_1$, Inequality (5) is implied by Inequality (4). This means that if v is unmatched, the only condition for $\{u,v\}$ to be a blocking pair is that u should have net increase in utility by getting matched with v. This is true even if v and w are neighbors. Canceling the factor of $1+\alpha_1$, we can thus summarize this (necessary and sufficient) condition for swivel to be an improving deviation as:

$$r_{uv} > r_{uw} \tag{6}$$

All improving deviations by a blocking pair can be classified as either a biswivel or a swivel, depending only on whether both nodes are matched or not. The following observation will later be useful. It is straightforward with inequalities (2) and (3) for a biswivel and inequality (6) for a swivel.

Observation 1. *Suppose nodes u and w are matched in M. If $\{u,v\}$ forms a blocking pair, then $r_{uv} > r_{uw}$.*

2.1 Existence and Welfare of Stable Matchings with Friendship Utilities

Theorem 1. *A stable matching exists in stable matching games with equal sharing and friendship utilities. Moreover, the set of stable matchings without friendship (i.e., when $\boldsymbol{\alpha} = \mathbf{0}$) is a subset of the set of stable matchings with friendship utilities on the same graph.*

Proof. We know from [3] that a stable matching M exists in the special case of correlated stable matching without friendship utilities, i.e., when $\boldsymbol{\alpha} = \mathbf{0}$. Now we prove that the same matching M is stable even when we have friendship utilities.

Suppose for contradiction that M is unstable for some value of $\boldsymbol{\alpha}$. This is possible only if we have a blocking pair $\{u, v\}$. But this cannot happen because:

- If both u and v were unmatched in M then M could not have been stable for $\boldsymbol{\alpha} = \mathbf{0}$.
- If exactly one of u and v is unmatched in M, say u is matched to w and v is unmatched, then for $\{u, v\}$ to be a blocking pair, $r_{uv} > r_{uw}$ by Observation 1. But in such a case, M could not have been stable for $\boldsymbol{\alpha} = \mathbf{0}$.
- Suppose both u and v are matched in M, say u is matched to w and v is matched to z. In such a case if $\{u, v\}$ forms a blocking pair corresponding to a biswivel, then by Observation 1, we have $r_{uv} > r_{uw}$ and $r_{uv} > r_{vz}$ and thus M could not have been stable for $\boldsymbol{\alpha} = \mathbf{0}$.

Hence we have shown that no blocking pair exists in M with friendship utilities, thus proving the theorem. □

Let us quickly comment on coalitional deviations. Stable matchings exactly compose the core of the stable matching game, i.e., every stable matching is also resilient to coalitional deviations that allow each deviating player to improve strictly. With externalities, however, this relation breaks. In particular, while the set of stable matchings (resilient to blocking pairs) expands when friendship increases, the core can become empty.

Example 4. As shown in Fig. 5, consider a game with 6 nodes $V = \{v_1, \ldots, v_6\}$. There is a square: $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$, $e_3 = \{v_3, v_4\}$, $e_4 = \{v_4, v_1\}$. In addition, there are two leaves: $e_5 = \{v_1, v_5\}$, $e_6 = \{v_3, v_6\}$. The rewards are $r_{e_1} = r_{e_3} = 1$, $r_{e_2} = r_{e_4} = 0$, $r_{e_5} = r_{e_6} = 1.1$. Consider $\alpha_1 = 1$ and $\alpha_d = 0$ for $d > 1$. Observe that v_1 and v_5 always have an incentive to deviate to e_5 as this swivel improves both their (perceived) rewards. The same holds for v_3 and v_6. Hence, the unique stable matching is $M = \{e_5, e_6\}$. However, in M the coalition $\{v_1, \ldots, v_4\}$ can then jointly deviate to $\{e_1, e_3\}$, increasing their perceived utility from 2.2 to 3. This shows that there is no strong equilibrium, i.e., the core of the game with friendship is empty. ■

While the core might be empty, our previous existence argument for stable matchings can be significantly generalized to the case of general symmetric, non-negative influence values $\alpha_{uv} = \alpha^u_{uv} = \alpha^v_{uv} \geq 0$. Let us define $q_{uv} =$

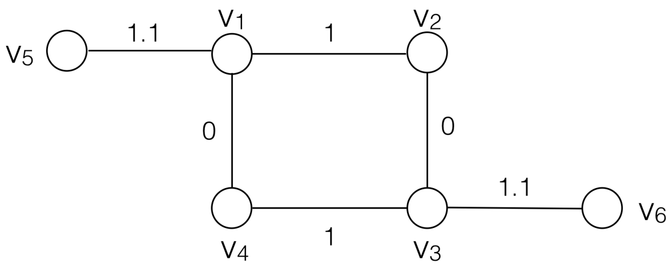

Fig. 5 Example in which a strong equilibrium does not exist with friendship utilities

$(1+\alpha_{uv})r_{uv}$ for each unordered pair $u, v \in V$. We term q_{uv} the "perceived edge reward" for every edge $\{u, v\} \in E$. It is straightforward to set up the conditions for profitable biswivels and swivels similar to (2)-(5) above. By inspection, we see that necessary conditions for a profitable biswivel are $q_{uv} > q_{uw}$ and $q_{uv} > q_{vz}$. Similarly, for a swivel it is necessary that $q_{uv} > q_{uw}$. This implies the same statement as in Observation 1 using q_{uv} instead of r_{uv}. The perceived edge reward is the same for both incident players. Now define an auxiliary stable matching game SMG_q with equal sharing and no friendship, which has exactly the same set of edges, and each edge $\{u, v\} \in E$ has reward q_{uv}. A stable matching M exists in SMG_q. As we only strengthen the requirements for a blocking pair when going from SMG_q to our game with friendship, M is also stable in our game with friendship.

Corollary 1. *A stable matching exists in stable matching games with equal sharing and friendship utilities based on general symmetric, non-negative α.*

As the last result in this section, we bound the price of anarchy.

Theorem 2. *The price of anarchy in stable matching games with equal sharing and friendship utilities is at most 2, and this bound is tight.*

Proof sketch. The idea of the proof is the following. Denote an optimal solution and a stable matching by M^* and M respectively. Let x^*_{uv} and x_{uv} denote the fraction with which edge $\{u, v\}$ is present in M^* and M respectively. Consider an edge $\{u, v\}$ such that $x^*_{uv} > x_{uv}$. If the fraction of the edge $\{u, v\}$ in M was increased to x^*_{uv} from x_{uv} by decreasing fractions of some other edges in M incident on u and v, then at least one of the endpoints of $\{u, v\}$ does not improve its utility. Tag this endpoint as corresponding to edge $\{u, v\}$ and denote the set of tagged nodes by B. We get one inequality for each node $u \in B$ for such a modification of fractions of adjoining edges. The critical step is to add all such inequalities to obtain the following:

$$\sum_{\substack{\{u,v\} \\ \text{s.t. } x^*_{uv} > x_{uv}}} q_{uv} \cdot (x^*_{uv} - x_{uv}) \leq 2 \cdot \sum_{\substack{\{u,v\} \\ \text{s.t. } x^*_{uv} < x_{uv}}} q_{uw} \cdot (x_{uw} - x^*_{uw}) \ . \quad (7)$$

The above inequality bounds the contribution made by the edges with a "stronger presence" in M^* in terms of the contribution made by the edges with a stronger presence in M. Simple algebraic manipulations along with adding the contribution from the remaining edges (with $x^*_{uv} = x_{uv}$) leads us to proving our result:

$$\sum_{\{u,v\}} q_{uv} x^*_{uv} \leq 2 \cdot \sum_{\{u,v\}} q_{uv} x_{uv}$$

We give an intuition for Equation 7 by drawing analogy to the widely-known result that a maximum matching in a graph has at most twice the number of edges of a maximal matching. While proving this result, the common edges are discounted and later the fact is used that each edge in a maximum matching (which is not present in a maximal matching), is accounted by at least one adjacent edge which is present in the maximal matching (and this edge is absent in the maximum matching). The factor 2 results because each edge present in a maximal matching can be used to account for two adjoining edges in a maximum matching. Equation 7 extends this technique to the case of fractionally stable matchings. □

We omit the detailed proof of Theorem 2 because its proof can be obtained as a special case of the proof of our much more general Theorem 6 presented later, where we show a bound on the price of anarchy of $1 + \frac{R+\alpha_1}{1+\alpha_1 R}$, with R being a measure of how unequally players can share rewards on an edge. When players share edge rewards equally, the bound on the price of anarchy in Theorem 6 reduces to $1 + \frac{1+\alpha_1}{1+\alpha_1} = 2$, as desired. Observe that Theorem 6 provides this bound even for all fractional stable matchings. We have seen in Example 1 in the introduction that this bound is tight, and in Example 3 that it does not extend to friendship utilities based on general symmetric, non-negative α.

2.2 Price of Stability and Convergence

The main result in this section bounds the price of stability in stable matching games with friendship utilities to $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$, and this bound is tight (see Theorem 4 below). This bound has some interesting implications. It is decreasing in each of α_1 and α_2, indicating that having friendship utilities (i.e., caring about the rewards of your friends more, and thus having higher α_1 or α_2 values) results in lower price of stability than without friendship utilities (i.e., when all α values equal 0). Also, note that values of $\alpha_3, \alpha_4, ..., \alpha_{diam(G)}$ have no influence on this bound. Thus, caring about players more than distance 2 away does not improve the price of stability in any way. Also, if $\alpha_1 = \alpha_2 = 1$, then PoS = 1, i.e., there will exist a stable matching which will also be a social optimum. Thus *loving thy neighbor and thy neighbor's neighbor but nobody beyond* is sufficient to guarantee that there exists at least one socially optimal stable matching. In fact, due to the shape of the curve, even small values of

```
1. Initialize M = M* where M* is a socially optimum matching.
2. If there is no relaxed blocking pair, terminate. Otherwise, resolve relaxed
   blocking pair {u,v} with maximum edge reward r_uv by adding {u,v} to M and
   removing from M any other matching edges incident to u and v.
3. Repeat step 2.
```

Fig. 6 Best-Relaxed-Blocking-Pair Algorithm

friendship quickly decrease the price of stability; e.g., setting $\alpha_1 = \alpha_2 = 0.1$ already decreases the price of stability from 2 to ~ 1.7.

We will establish the bound on the price of stability by defining an algorithm that creates a good stable matching in polynomial time. One possible idea to create a stable matching that is close to optimum is to use a Best-Blocking-Pair algorithm: Start with the best possible matching, i.e. a social optimum, which may or may not be stable. Now choose the "best" blocking pair $\{u, v\}$, the one with maximum edge reward r_{uv}. Allow this blocking pair to get matched to each other instead of their current partners. Check if the resulting matching is stable. If it is not stable, then allow the best blocking pair for this matching to get matched. Repeat the procedure until there are no more blocking pairs, thereby obtaining a stable matching.

This algorithm gives the desired price of stability and running time bounds for the case of "altruism" when all α_i are the same, see Corollary 2 below. To provide the desired bound with general friendship utilities, we must alter this algorithm slightly using the concept of *relaxed* blocking pair.

Definition 3. *Given a matching M, we call a pair of nodes $\{u, v\}$ a relaxed blocking pair if either $\{u, v\}$ form an improving swivel, or u and v are matched to w and z respectively, with the following inequalities being true:*

$$(1 + \alpha_1)r_{uv} > (1 + \alpha_1)r_{uw} + (\alpha_1 + \alpha_2)\, r_{vz} \tag{8}$$

$$(1 + \alpha_1)r_{uv} > (1 + \alpha_1)r_{vz} + (\alpha_1 + \alpha_2)\, r_{uw} \tag{9}$$

In other words, a relaxed blocking pair ignores the possible edges between nodes u and z, and has α_2 in the place of α_{uz} (similarly, α_2 in the place of α_{vw}). It is clear from this definition that a blocking pair is also a relaxed blocking pair, since the conditions above are less constraining than Inequalities (2) and (3). Thus a matching with no relaxed blocking pairs is also a stable matching. Moreover, it is easy to see that Observation 1 still holds for relaxed blocking pairs. We will call a relaxed blocking pair satisfying Inequalities (8) and (9) a *relaxed biswivel,* which may or may not correspond to an improving deviation, since a relaxed blocking pair is not necessarily a blocking pair.

2.2.1 The Best-Relaxed-Blocking-Pair *Algorithm*

We use the Best-Relaxed-Blocking-Pair algorithm shown in Figure 6 to compute a near-optimal stable matching. To establish the efficient running time and the bound on the price of stability of the resulting stable matching, we first analyze the dynamics of this algorithm and prove some helpful lemmas. We can interpret the algorithm as a sequence of swivel and relaxed biswivel deviations, each inserting one edge into M, and removing up to two edges. It is not guaranteed that the inserted edge will stay forever in M, as a subsequent deviation can remove this edge from M. Let $O_1, O_2, O_3, \cdots$ denote this sequence of deviations, and $e(i)$ denote the edge which got inserted into M because of O_i. We analyze the dynamics of the algorithm as follows.

Observation 2. *During the execution of* Best-Relaxed-Blocking-Pair, *the first deviation O_1 is a relaxed biswivel.*

This is straightforward, as O_1 being a swivel would strictly improve the value of the matching by Observation 1. As we begin the algorithm with $M = M^*$, O_1 cannot be a swivel, because there is no matching with value strictly greater than M^*.

Lemma 1. *Let O_j be a relaxed biswivel that takes place during the execution of the best relaxed blocking pair algorithm. Suppose a deviation O_k takes place before O_j. Then we have $r_{e(k)} \geq r_{e(j)}$. Furthermore, if O_k is a relaxed biswivel then $e(k) \neq e(j)$ (thus at most $|E(G)|$ relaxed biswivels can take place during the execution of the algorithm).*

It is important to note that this lemma does *not* say that $r_{e(i)} \geq r_{e(j)}$ for $i < j$. We are only guaranteed that $r_{e(i)} \geq r_{e(j)}$ for $i < j$ if O_j is a *relaxed biswivel.* Between two successive relaxed biswivels O_k and O_j, the sequence of $r_{e(i)}$ for consecutive swivels can and does increase as well as decrease, and the same edge may be added to the matching multiple times. All that is guaranteed is that $r_{e(j)}$ for a biswivel O_j will have a lower value than all the preceding $r_{e(i)}$'s. Thus, this lemma suggests a nice representation of Best-Relaxed-Blocking-Pair in terms of phases, where we define a *phase* as a subsequence of deviations that begins with a relaxed biswivel and continues until the next relaxed biswivel. Observation 2 shows that the start of the sequence is also the start of the first phase. Lemma 1 guarantees that at the start of each phase, the $r_{e(j)}$ value is smaller than the values in all previous phases, and that there is only a polynomial number of phases. Now we proceed to prove Lemma 1.

Proof. Let $e(j) = \{v, z\}$ get inserted in M because of a relaxed biswivel O_j. We first give a brief outline of the proof. Suppose that the claim $r_{e(k)} \geq r_{e(j)}$ for $k < j$ is false, and we have an O_k with $k < j$ such that $r_{e(k)} < r_{e(j)}$. Clearly $\{v, z\}$ could not have been a relaxed blocking pair just before O_k, otherwise the algorithm would have chosen $\{v, z\}$ as the best relaxed blocking pair instead of O_k. We will show that this leads to a conclusion that $\{v, z\}$ cannot be a relaxed blocking pair even for O_j. This is a contradiction, hence our assumption of $r_{e(k)} < r_{e(j)}$ could not have been correct. Thus for all O_k such that $k < j$ we will have $r_{e(k)} \geq r_{e(j)}$. Later we will use similar reasoning to prove that if O_i with $i < j$ is a relaxed biswivel that takes place before a relaxed biswivel O_j, then $e(i) \neq e(j)$. Now let us proceed to the proof.

Suppose to the contrary that we have O_k with $k < j$ such that $r_{e(k)} < r_{e(j)}$ with O_j being a relaxed biswivel. As discussed in the outline of the proof, this implies that $\{v, z\}$ was not a relaxed blocking pair at the time O_k was selected. Let S be the set of nodes with whom v and z are matched at the time that O_k is selected. As long as S does not change, v and z will not be a relaxed blocking pair, since the change in utility experienced by v and z from matching to each other depends only on their partners in the current matching, i.e., the set S. Thus for the relaxed biswivel O_j to occur, S must change between O_k and O_j. We will show that this leads to a contradiction: $\{v, z\}$ cannot be a relaxed blocking pair at the time O_j is selected.

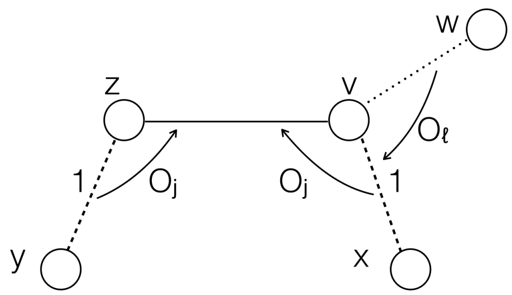

Fig. 7 Illustration for the proof of Lemma 1

Suppose v is matched to x and z is matched to y just before biswivel O_j (see Figure 7). Since $\{v, z\}$ is a relaxed blocking pair at this point, we thus have

$$(1 + \alpha_1) r_{vz} > (1 + \alpha_1) r_{vx} + (\alpha_1 + \alpha_2) r_{zy} \tag{10}$$
$$(1 + \alpha_1) r_{vz} > (1 + \alpha_1) r_{zy} + (\alpha_1 + \alpha_2) r_{vx} \ . \tag{11}$$

Recall that $\{v, z\}$ was not a relaxed blocking pair just before O_k, and to make it a relaxed blocking pair for O_j, S must change between O_k and O_j. Let O_l be the last deviation which changed S to $\{x, y\}$. Without loss of generality, we can assume that O_l adds the edge $\{v, x\}$. Now we have two cases:

- $\{v, z\}$ was a relaxed blocking pair at the time O_l is selected. In this case $\{v, x\}$ could not have been the best relaxed blocking pair for O_l, because inequality (10) tells us $r_{vz} > r_{vx}$.
- $\{v, z\}$ was not a relaxed blocking pair at the time O_l is selected. Suppose v was matched with w before O_l. As $\{v, z\}$ was not a relaxed blocking pair just before O_l, we have

$$\text{Either } (1 + \alpha_1) r_{vz} \le (1 + \alpha_1) r_{vw} + (\alpha_1 + \alpha_2) r_{zy} \tag{12}$$
$$\text{OR } (1 + \alpha_1) r_{vz} \le (1 + \alpha_1) r_{zy} + (\alpha_1 + \alpha_2) r_{vw} \ . \tag{13}$$

(If v was unmatched just before O_l, then substitute $r_{vw} = 0$ to obtain the appropriate condition.) Assume that it is inequality (12) that holds. Then, because O_l removes edge $\{v, w\}$ and adds edge $\{v, x\}$, we have $r_{vx} > r_{vw}$ as Observation 1 holds for relaxed blocking pairs. Thus, it holds

$$(1 + \alpha_1)r_{vz} \leq (1 + \alpha_1)r_{vx} + (\alpha_1 + \alpha_2)r_{zy} \ . \tag{14}$$

This contradicts inequality (10), and thus $\{v, z\}$ cannot be a relaxed blocking pair at the time O_j is selected. The same conclusion can be reached if we assume inequality (13) holds true.

In either case we obtain a contradiction, thus showing that if O_j is a relaxed biswivel, then for all O_k with $k < j$, we have $r_{e(k)} < r_{e(j)}$.

The only remaining piece is to prove $e(k) \neq e(j)$ if O_k is a relaxed biswivel. Notice that if $e(k) = e(j) = \{v, z\}$, then S has to change between O_k and O_j. Now we use exactly the reasoning from the previous paragraph to arrive at a contradiction, thus proving that $e(k) \neq e(j)$. □

If $\alpha_1 = \alpha_2$, the conditions for a blocking pair are identical to the conditions for a relaxed blocking pair. Hence, our algorithm corresponds to letting the best blocking pair deviate at each step. As a special case, for $\boldsymbol{\alpha} = \mathbf{0}$ and correlated stable matching, this algorithm is known to provide a stable matching in polynomial time [4]. For friendship utilities, however, (quick) convergence was previously unknown. We show that even with the addition of friendship, Best-Relaxed-Blocking-Pair (and thus Best-Blocking-Pair when $\alpha_1 = \alpha_2$) terminates and produces a stable matching in polynomial time.

If instead of the best we pick some arbitrary blocking pair, then there exists an instance in which, starting from the empty matching, a sequence of blocking pairs of length $2^{\Omega(n)}$ exists until reaching a stable matching, even without friendship. This is directly implied by recent results in correlated stable matching [26].[1]

Moreover, it was shown recently in [33] that there is also a sequence of blocking pairs of polynomial length until reaching a stable matching with friendship utilities. However, [33] does not consider Best-Relaxed-Blocking-Pair at all, and the sequence is not necessarily the one computed by Best-Blocking-Pair. Also, in contrast to our case, the resulting matching in [33] does not yield guarantees on the social welfare, which we show below in the next section.

Theorem 3. Best-Relaxed-Blocking-Pair *outputs a stable matching after $O(m^3)$ iterations, where m is the number of edges in the graph.*

[1] A trivial adjustment of the gadget in [26] allows us to construct the exponential sequence even when starting from the social optimum. We scale the reward of each (original) edge $i \in \{1, \ldots, m\}$ in the gadget from i to $1 + i \cdot \epsilon$, for some tiny $\epsilon > 0$. This preserves all incentives and the structure of all blocking pairs. Then, we add an auxiliary neighbor for each (original) player and connect it via an auxiliary edge of reward 1. The social optimum is obviously given by matching each original player with his auxiliary neighbor. However, the exponential sequence of blocking pairs still exists, because auxiliary edges are not rewarding enough to influence blocking pairs among original players.

Proof. Consider the three possible changes that can occur to the matching M during each iteration: a swivel could add a new edge, or it could delete an edge and add an edge with strictly higher r_e value. A relaxed biswivel deletes two edges and adds an edge with higher r_e value than either. If no biswivels takes place, and instead only swivel deviations take place, then the number of edges in the matching cannot decrease. Also observe that swivel deviations cannot form a cycle, i.e., if edge e_1 is removed by a swivel and edge e_2 is added, and then later e_2 is removed and e_3 is added, and so on until some edge e_k, then it cannot be that $e_k = e_1$. Such a cycle would lead to a contradiction since $r_{uv} > r_{uw}$ if a swivel causes removal of edge $\{u, w\}$ to insert edge $\{u, v\}$. Since the number of edges in a matching cannot decrease with swivel deviations and swivels cannot form a cycle, it implies that the maximum number of consecutive swivels that can take place is $O(m^2)$.

Now by Lemma 1 there are at most m relaxed biswivel deviations, so the algorithm terminates after $O(m^3)$ deviations. Since there are no more relaxed blocking pairs for the algorithm to continue, and since a blocking pair is also a relaxed blocking pair, the final matching produced by the algorithm is a stable matching. □

2.2.2 Upper Bound on the Price of Stability

As seen above, during the execution of Best-Relaxed-Blocking-Pair algorithm we can have only a polynomial number of consecutive swivel deviations between each relaxed biswivel. We also know that every phase (defined as a maximal subsequence of consecutive swivels) lasts only a polynomial amount of time, and there are only $O(m)$ phases by Lemma 1. Moreover, in each phase, the value of the matching only increases, since swivels only remove an edge if they add a better one. Thus only relaxed biswivels operations can reduce the cost of the matching during the execution of Best-Relaxed-Blocking-Pair algorithm. We use these properties below to bound the cost of the stable matching this algorithm produces.

To prove the bound, we will need some notation. We define a sequence of mappings from M^* to $E(G)$. Define $h_0 : M^* \to E(G)$ as $h_0(e) = e$. Depending on O_i, we define h_i as follows: Suppose O_i is a deviation that removes edge $h_{i-1}(e_j)$ from M. If O_i inserts edge e_l in M then set $h_i(e_j) = e_l$. For all other $e_k \in M^*$, keep $h_i(e_k)$ same as $h_{i-1}(e_k)$. Let us note that a deviation O_i may not remove any edges from $\{h_{i-1}(e_j) : e_j \in M^*\}$. This can happen because during the course of the algorithm, two unmatched nodes can get matched, say to insert e_p into M. No edges in M^* get mapped to e_p. If this edge is removed from M by a later deviation, the mapping may not change, since no edge is mapped to e_p. To summarize, h_i may be the same as h_{i-1}, or may differ from h_{i-1} in one location (in case of a swivel), or in two locations (in case of a relaxed biswivel). Denote the resulting mapping when our algorithm terminates by h_M.

Intuitively, think of the functions h_i as labels for each edge in the current matching. The set of possible labels is that of the edges in M^*. At the start, each edge of M^* is labeled with itself. At the end, $h_M(e)$ is the edge of M labeled with the label e. Our algorithm "converts" edge e into $h_M(e)$ during its entire execution. Note that an edge of M can have many labels at the same time, since after a biswivel in which e_1 and e_2 are removed and e_3 is added, e_3 gets the labels of both e_1 and e_2.

Coupling Observation 1 with the definition of mappings h_i, we get:

Observation 3. *$\{r_{h_i(e)}\}_{i\geq 0}$ is a nondecreasing sequence and $r_{h_{i+1}(e)} > r_{h_i(e)}$ whenever $h_{i+1}(e) \neq h_i(e)$.*

The observation holds because the only reason for $r_{h_{i+1}(e)} \neq r_{h_i(e)}$ is when the deviation O_i causes a node to obtain higher utility by switching to $h_{i+1}(e)$ from $h_i(e)$ (either due to a biswivel or a swivel). Then Observation 1 states that the edge reward of the new edge $r_{h_{i+1}(e)}$ must be greater than $r_{h_i(e)}$. Now the next lemma will be instrumental in proving the bound on the price of stability.

Lemma 2. *If $h_M(e_i) = h_M(e_j)$ with $e_i \neq e_j$ then*

1. *There must exist a relaxed biswivel O_k such that $h_{k-1}(e_i) \neq h_{k-1}(e_j)$ but O_k makes $h_k(e_i) = h_k(e_j)$. Furthermore, for all $p \geq k$ we have $h_p(e_i) = h_p(e_j)$.*
2. *There does not exist another $e_l \in M^*$ such that $h_M(e_l) = h_M(e_i) = h_M(e_j)$.*
3. *$r_{e_i} + r_{e_j} < \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot r_{h_M(e_i)}$*

Proof. To prove the first part, say O_l was the first deviation such that $h_{l-1}(e_i) \neq h_{l-1}(e_j)$ and $h_l(e_i) = h_l(e_j)$. This cannot happen because of a swivel deviation because a swivel can make $h_l(e) \neq h_{l-1}(e)$ for at most for one $e \in M^*$. Thus O_l must be a relaxed biswivel. Set $k = l$, and it is easy to see that for $p \geq k$ we have $h_p(e_i) = h_p(e_j)$. Hence the first part is proven.

To prove the second part, suppose there exists an e_l with $e_l \neq e_i \neq e_j$ such that $h_M(e_l) = h_M(e_i) = h_M(e_j)$. From the first part, there must exist a relaxed biswivel O_k s.t. $h_{k-1}(e_i) \neq h_{k-1}(e_l)$ but $h_k(e_i) = h_k(e_l)$. Similarly there must exist a relaxed biswivel O_p s.t. $h_{p-1}(e_i) \neq h_{p-1}(e_j)$ but $h_p(e_i) = h_p(e_j)$. Without loss of generality say $p > k$. Using Lemma 1 we get $r_{e(k)} \geq r_{e(p)}$. But from Observation 3, we have $r_{e(k)} < r_{e(p)}$, since $e(p) = h_p(e_i) \neq h_k(e_i) = e(k)$. We obtain a contradiction here, thus proving that there does not exist another $e_l \in M^*$ with $h_M(e_l) = h_M(e_i) = h_M(e_j)$.

To prove the third part, consider a relaxed biswivel O_k such that $h_{k-1}(e_i) \neq h_{k-1}(e_j)$ and $h_k(e_i) = h_k(e_j)$. Substitute $r_{uv} = r_{h_k(e_i)}$, $r_{uw} = r_{h_{k-1}(e_i)}$ and $r_{vz} = r_{h_{k-1}(e_j)}$ in inequalities (2) and (3). Adding these inequalities and simplifying we get

$$r_{h_{k-1}(e_i)} + r_{h_{k-1}(e_j)} < \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot r_{h_k(e_i)} \ . \tag{15}$$

From Observation 3, we know $\{r_{h_i(e)}\}_{i\geq 0}$ is a nondecreasing sequence. Using this in (15) we get

$$r_{e_i} + r_{e_j} < \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot r_{h_M(e_i)} \ . \tag{16}$$

□

Lemma 2 states that for every edge $e \in M$, at most two edges of M^* can map to it, i.e., there are at most two edges e_i and e_j such that $h_M(e_i) = h_M(e_j) = e$. Let B denote the set of edges $e_i \in M^*$ such that $h_M(e_i) = h_M(e_j)$ for some other $e_j \in M^*$. These are the edges which "share" the edge e of M they are mapped to because another edge of M^* is mapped to the same edge e. Let A denote the remaining edges in M^*, i.e., the edges $e_i \in M^*$ so that no other edge of M^* is mapped to the same value $h_M(e_i)$. We can further partition set B into two sets P and Q as follows. B naturally consists of pairs of edges e_i and e_j of M^* which have the same value $h_M(e_i) = h_M(e_j)$. We will take one of these edges from each such pair and put it into P, while we put the other one into Q. More formally, take one such pair e_i and e_j and choose one of these two edges arbitrarily (say we choose e_i). Add e_i to P, and the other edge in the pair (which we will denote as $\mu(e_i)$) into Q. The matching M computed by Best-Relaxed-Blocking-Pair has value at least $\sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} r_{h_M(e)}$, since all of the edges of A and P are mapped to unique edges of M. Possible additional edges in M may also exist because of swivels which match two previously unmatched nodes with each other, but this is certainly a lower bound on the value of M.

This allows us to prove the main theorem of this section.

Theorem 4. *The price of stability in stable matching games with equal sharing and friendship utilities is at most $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$, and this bound is tight.*

Proof. The value of M^* is given by

$$w(M^*) = \sum_{e \in A} r_e + \sum_{e \in P} r_e + \sum_{e \in Q} r_e = \sum_{e \in A} r_e + \sum_{e \in P} (r_e + r_{\mu(e)}) \ .$$

Using Lemma 2, for $e \in P$ we have $r_e + r_{\mu(e)} < \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot r_{h_M(e)}$. Using Observation 3, for $e \in A$ we have $r_e \leq r_{h_M(e)}$. Thus, we get

$$w(M^*) \leq \sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot r_{h_M(e)} \leq \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot \left(\sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} r_{h_M(e)} \right) \ .$$

Note that this inequality may not be strict since A may be empty. This could happen if each edge in M^* gets removed because of a relaxed biswivel as the algorithm proceeds (though it may be possible that it is inserted later). We also have $w(M) \geq \sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} r_{h_M(e)}$ for the final matching M of the algorithm. Using this,

$$w(M^*) \leq \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot w(M) \ ,$$

which proves the bound on the price of stability, since M is a stable matching.

To prove the tightness of the bound, assume $\alpha_2 = 0$ and set $r_{uv} = \frac{1+2\alpha_1+\epsilon}{1+\alpha_1}$, $r_{uw} = r_{vz} = 1$ in Fig 3. Then we have $\{\{u, v\}\}$ as the only stable matching but the social optimum is $\{\{u, w\}, \{v, z\}\}$. Thus, we get a price of stability of $\frac{2+2\alpha_1}{1+2\alpha_1+\epsilon}$. With $\epsilon \to 0$, this yields a tight bound for $\alpha_2 = 0$. □

Theorems 3 and 4, yield the following corollary about the behavior of best blocking pair dynamics. It applies in particular to the model of altruism when $\alpha_i = \alpha$ for all $i = 1, \ldots, diam(G)$, as for $\alpha_1 = \alpha_2$, Best-Relaxed-Blocking-Pair is Best-Blocking-Pair.

Corollary 2. *If $\alpha_1 = \alpha_2$ and we start from a socially optimum matching,* Best-Blocking-Pair *converges in $O(m^2)$ time to a stable matching that is at most a factor of $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$ worse than the optimum.*

3 Stable Matching with Friendship and General Reward Sharing

In the previous section we assumed that for $\{u, v\} \in M$ both u and v get the *same reward* r_{uv}. Let us now treat the more general case where u and v receive different rewards for $\{u, v\} \in M$. We define r^x_{xy} as the reward of x from edge $\{x, y\} \in M$. We interpret our model in a reward sharing context, where x and y share a total reward of $r_{xy} = r^x_{xy} + r^y_{xy}$. The correlated matching model of Section 2 can equivalently be formulated as equal sharing with nodes u and v receiving a reward of $r_{uv}/2$.

Let us again write explicit conditions for nodes to form a blocking pair in this context and define some helpful notation. The necessary and sufficient conditions for nodes $\{u, v\}$ to form a biswivel from nodes w and z (See Fig. 3) in reward sharing with friendship are

$$\begin{aligned} r^u_{uv} + \alpha_1 r^v_{uv} &> r^u_{uw} + \alpha_1(r^w_{uw} + r^v_{vz}) + \alpha_{uz} r^z_{vz} \\ r^v_{uv} + \alpha_1 r^u_{uv} &> r^v_{vz} + \alpha_1(r^z_{vz} + r^u_{uw}) + \alpha_{vw} r^w_{uw} \ . \end{aligned}$$

We define $q^x_{xy} = r^x_{xy} + \alpha_1 r^y_{xy}$. Then the conditions for biswivel such as shown in Fig. 3 are

$$q^u_{uv} > q^u_{uw} + \alpha_1 r^v_{vz} + \alpha_{uz} r^z_{vz} \quad (17)$$

$$q^v_{uv} > q^v_{vz} + \alpha_1 r^u_{uw} + \alpha_{vw} r^w_{uw} \ . \quad (18)$$

Similarly, the necessary and sufficient conditions for swivel (See Fig. 4) are

$$\begin{aligned} r^u_{uv} + \alpha_1 r^v_{uv} &> r^u_{uw} + \alpha_1 r^w_{uw} \\ r^v_{uv} + \alpha_1 r^u_{uv} &> \alpha_1 r^u_{uw} + \alpha_{vw} r^w_{uw} \ . \end{aligned}$$

Using the definition of q^x_{xy}, the conditions for swivel become

$$q^u_{uv} > q^u_{uw} \quad (19)$$

$$q^v_{uv} > \alpha_1 r^u_{uw} + \alpha_{vw} r^w_{uw} \ . \quad (20)$$

Let us define $q_{xy} = q^x_{xy} + q^y_{xy}$. Thus we obtain $q_{xy} = (1 + \alpha_1) r_{xy}$.

3.1 Existence of a Stable Matching

Without friendship utilities, our stable matching game reduces to the stable roommates problem (i.e., non-bipartite stable matching), since reward shares can be arbitrary and thus induce arbitrary preference lists for each node. It is well known that a stable matching may not exist in instances of the stable roommates problem [23]. While we are able to prove existence of integral stable matching for several interesting special cases (see Section 3.4 below), the addition of friendship further complicates matters. In Section 2.1 we showed that for equal sharing, a stable matching without friendship utilities (i.e., $\boldsymbol{\alpha} = \mathbf{0}$) is also a stable matching when we have friendship utilities. This is no longer true for unequal reward sharing: adding a social context can completely change the set of stable matchings. To see this, consider the following example, where adding a social context (i.e., increasing $\boldsymbol{\alpha}$ above zero) destroys all stable matchings that exist when $\boldsymbol{\alpha} = \mathbf{0}$. In fact, Example 5 goes even further to show that the set of stable matchings with and without friendship could be completely non-overlapping.

Example 5. Consider the graph on the left hand side from Fig. 4. We assign $r^u_{uw} = r^w_{uw} = 1$, $r^u_{uv} = 10/11$, $r^v_{uv} = 100/11$ with $\alpha_1 = 1/2$ and $\alpha_2 = \alpha_3 = \cdots = 0$. Without friendship utilities, $\{\{u, w\}\}$ is the only stable matching as u and w will always want to get matched to each other. However, with friendship utilities we have $q^u_{uv} = \frac{60}{11}$, $q^u_{uw} = \frac{3}{2}, q^v_{uv} = \frac{105}{11}$, $q^v_{uw} = \frac{3}{2}$. Thus, using inequalities (19) and (20) we see that with friendship utilities, the only stable matching is $\{\{u, v\}\}$ as u will always want to get matched to v. Thus for unequal reward sharing with friendship utilities, the set of stable matchings can be completely non-overlapping with the set of stable matchings for unequal reward sharing without friendship utilities. ■

Although stable matchings may not exist in general non-bipartite graphs, *fractional* stable matchings are guaranteed to exist [2]. Fortunately, as we prove below, this holds even in the presence of friendship utilities with general reward sharing: A fractional stable matching always exists.

A fractional stable matching is a fractional matching without blocking pairs. Specifically, a biswivel occurs when there is an edge $\{u, v\}$ such that increasing the strength of edge $\{u, v\}$, and decreasing the strength of some other edges $\{u, w\}$ and $\{v, z\}$ would strictly improve the utilities of both u and v. The inequalities that would make this be true are exactly (17) and (18); they do not change simply due to the fractional nature of the matching (note, however, that for a biswivel to make sense, it is necessary that $x_{uv} < 1$, $x_{uw}, x_{vz} > 0$). Similarly, a swivel occurs when increasing the strength of an edge $\{u, v\}$ with node v not being tight (i.e., $\sum_{e \ni v} x_e < 1$), and decreasing the strength of some edge $\{u, w\}$ would strictly improve the utilities of both u and v; or when there are two adjacent nodes that are not tight. The inequalities that would make this be true are exactly (19) and (20).

Theorem 5. *A fractional stable matching always exists, even for general reward sharing and friendship utilities.*

Proof. We use the same line of reasoning as for equal sharing above. We denote by SMG_q an auxiliary stable matching game where we have exactly the same edges, no friendship, and rewards as in q, i.e., a node u will prefer node v over w iff $q^u_{uv} > q^u_{uw}$, breaking ties arbitrarily. SMG_q has at least one fractional (and, in fact, half-integral) stable matching [2]. Similar as before, the requirements for a blocking pair in our game with friendship are stronger than in the auxiliary game. Hence, a (fractional) stable matching in SMG_q remains stable in our game with friendship.

More formally, suppose a fractional stable matching M for SMG_q is not a fractional stable matching for unequal reward sharing with friendship utilities. Then there exists a blocking pair $\{u, v\}$ with one of the following two possibilities:

- $\{u, v\}$ forms a biswivel, so the inequalities (17) and (18) must hold true. These inequalities imply $q^u_{uv} > q^u_{uw}$ and $q^v_{uv} > q^v_{vz}$. But then $\{u, v\}$ would be a blocking pair in SMG_q. This contradicts that M is stable in SMG_q.
- $\{u, v\}$ forms a swivel, say with v such that $\sum_{e\ni v} x_e < 1$ and with u such that $\sum_{e\ni u} x_e = 1$. (It cannot be that both u and v are not tight, since otherwise M would not be stable in SMG_q.) Then for $\{u, v\}$ to be a blocking pair inequalities (19) and (20) must hold true. But these inequalities imply $q^u_{uv} > q^u_{uw}$ and thus $\{u, v\}$ would be a blocking pair in SMG_q. This contradicts that M is stable in SMG_q.

Hence, M must be stable with unequal reward sharing and friendship utilities. Moreover, the set of fractional stable matchings in SMG_q is a subset of the set of fractional stable matchings in unequal reward sharing with friendship utilities. Since there exists at least one fractional stable matching in SMG_q, the theorem is proved. □

Note that the argument extends to general non-negative values of α with $\alpha^u_{uv}, \alpha^v_{uv} \geq 0$. Setting up conditions for profitable biswivels and swivels, and defining $q^x_{xy} = r^x_{xy} + \alpha^x_{xy} r^y_{xy}$ allows to derive the necessary conditions $q^u_{uv} > q^u_{uw}$ and $q^v_{uv} > q^v_{vz}$ for profitable biswivel, and $q^u_{uv} > q^u_{uw}$ for swivel. Hence, using the relation to the auxiliary game SMG_q, we see that our game has at least one fractional stable matching.

Corollary 3. *A fractional stable matching always exists, even for general reward sharing and friendship utilities based on general non-negative* α.

3.2 Price of Anarchy with General Reward Sharing

In this section we prove tight bounds for the price of anarchy of stable matching with friendship utilities in the presence of general reward sharing. Since an integral stable matching may not exist, we instead consider fractional matching; by price of anarchy here we mean the ratio of the total reward in a socially optimum *fractional* matching with the worst *fractional* stable matching. The corresponding ratio between the integral versions is trivially upper bounded by this amount as well.

We define R as

$$R = \max_{\{u,v\} \in E(G)} \frac{r_{uv}^u}{r_{uv}^v} \ . \tag{21}$$

Note that we will always have $R \geq 1$. By definition of q, we also have

$$\frac{q_{xy}^x}{q_{xy}^y} = \frac{r_{xy}^x + \alpha_1 r_{xy}^y}{r_{xy}^y + \alpha_1 r_{xy}^x} \ .$$

Using the fact that $\frac{p+\alpha_1}{1+\alpha_1 p}$ is an increasing function of p and using the definition of R, we thus obtain

$$\frac{q_{xy}^x}{q_{xy}^y} \leq \frac{R + \alpha_1}{1 + \alpha_1 R} \ . \tag{22}$$

We show the following theorem.

Theorem 6. *The (fractional) price of anarchy for general reward sharing with friendship utilities is at most* $1 + Q$*, where* $Q = \max_{\{u,v\} \in E(G)} \frac{q_{uv}^u}{q_{uv}^v} \leq \frac{R+\alpha_1}{1+\alpha_1 R}$*, and this bound is tight.*

The proof of this theorem is very similar in spirit to the proof for the case without friendship (see, e.g., [8]) and is provided in Appendix A.1 for completeness. We can consider the quantity $q_{uv}^u = r_{uv}^u + \alpha_1 r_{uv}^v$ to be the "perceived edge reward" value of the edge (uv) for the node u, since it quantifies the contribution of edge (uv) to the utility of node u if u and v are matched. Then the ratio Q can be intuitively understood as the maximum disparity between the perceived edge reward values of an edge for its two endpoints.

Let us consider the implications of this bound. If $R = 1$, the bound is 2. This result implies Theorem 2, since when we have $R = 1$, then both u and v get the same reward from an edge $\{u, v\} \in M$. If $\alpha_1 = 0$, the bound is $1 + R$. The tightness of this bound implies that as sharing becomes more unfair, i.e., as $R \to \infty$, we can find instances where the price of anarchy is unbounded. Unequal sharing can make things much worse for the stable matching game.

Notice, however, that $\frac{R+\alpha_1}{1+\alpha_1 R}$ is a decreasing function of α_1. As α_1 increases from 0 to 1, the bound decreases from $1 + R$ to 2. Without friendship utilities ($\boldsymbol{\alpha} = \mathbf{0}$), we have a tight upper bound of $1+R$, which is extremely bad for large R. As α_1 tends to 1, however, the price of anarchy drops to 2, independent of R. For example, for $\alpha_1 = 1/2$ it is only 3. Thus, social context can drastically improve the outcome for the society, especially in the case of unfair and unequal reward sharing.

3.3 Price of Stability with General Reward Sharing

In this section, we give a simple lower bound $Q' = \frac{(1+\alpha_1)(1+R)}{1+\alpha_1(R+1)}$ on the price of stability for stable matching games with friendship and reward sharing. Furthermore, we show that this bound is within an addition of 1 to the optimum, i.e., $Q < Q' \leq \text{PoS} \leq 1 + Q$.

To prove the lower bound, we analyze the following example.

Example 6. Consider the 3-length path as shown in Fig. 3. Set $\alpha_2 = \alpha_3 = \cdots = 0$ and use the following rewards for an infinitesimal $\epsilon > 0$:

$$r^u_{uv} = \frac{1}{1+\alpha_1}\left(\frac{1+\alpha_1(R+1)}{(1+\alpha_1 R)} + \epsilon\right) \quad r^v_{uv} = \frac{1}{1+\alpha_1}\left(\frac{1+\alpha_1(R+1)}{(1+\alpha_1 R)} + \epsilon\right)$$
$$r^u_{uw} = \frac{1}{1+\alpha_1 R} \quad r^w_{uw} = \frac{R}{1+\alpha_1 R}$$
$$r^v_{vz} = \frac{1}{1+\alpha_1 R} \quad r^z_{vz} = \frac{R}{1+\alpha_1 R}$$

As desired we have $\max_{\{x,y\}\in E(G)} \frac{r^x_{xy}}{r^y_{xy}} = R$. Using $q^x_{xy} = r^x_{xy} + \alpha_1 r^y_{xy}$, we obtain

$$q^u_{uv} = \frac{1+\alpha_1(R+1)}{1+\alpha_1 R} + \epsilon \qquad q^v_{uv} = \frac{1+\alpha_1(R+1)}{1+\alpha_1 R} + \epsilon$$
$$q^u_{uw} = 1 \qquad q^w_{uw} = \frac{R+\alpha_1}{1+\alpha_1 R}$$
$$q^v_{vz} = 1 \qquad q^z_{vz} = \frac{R+\alpha_1}{1+\alpha_1 R}$$

As desired, we have $\max_{\{x,y\}\in E(G)} \frac{q^x_{xy}}{q^y_{xy}} = \frac{R+\alpha_1}{1+\alpha_1 R} = Q$. We have $\{\{u,v\}\}$ as a stable matching: $\{u,w\}$ is not a blocking pair, because $q^u_{uw} \leq q^u_{uv}$. Similarly $\{v,z\}$ will not be a blocking pair. Any other fractional matching is no longer stable because $\{u,v\}$ is a blocking pair as inequalities (17) and (18) are satisfied. However, $\{\{u,w\},\{v,z\}\}$ is still the socially optimal matching. Hence, the price of stability as $\epsilon \to 0$ is given by

$$\frac{r_{uw} + r_{vz}}{r_{uv}} = \frac{q_{uw} + q_{vz}}{q_{uv}} \to \frac{(1+\alpha_1)(1+R)}{1+\alpha_1(R+1)} \ .$$

■

Let us define $Q' = \frac{(1+\alpha_1)(1+R)}{1+\alpha_1(R+1)}$. The above example 6 establishes a lower bound on the price of stability of Q'. This along with the following theorem will prove that $Q < Q' \leq \text{PoS} \leq 1 + Q$.

Theorem 7. *The (fractional) price of stability of stable matching games with friendship and general reward sharing is in* $[Q', Q+1]$, *with* $Q < Q' \leq Q + 1$.

Proof. The only part that is yet to be proven is $Q \leq Q'$ and $Q' \leq 1 + Q$. We have

$$Q' - Q = \frac{(1 - \alpha_1 + \alpha_1 R)(1 + \alpha_1)}{(1 + \alpha_1 + \alpha_1 R)(1 + \alpha_1 R)} \ .$$

As $(1 - \alpha_1 + \alpha_1 R) \leq (1 + \alpha_1 + \alpha_1 R)$ and $1 + \alpha_1 \leq 1 + \alpha_1 R$, we have that $Q' - Q \leq 1$. As $R \geq 1$, the numerator is always positive. Hence $0 < Q' - Q \leq 1$. With our lower bound on the price of stability of Q', the theorem follows. □

3.4 Specific Reward Sharing Rules

In this section we consider some particularly natural reward sharing rules and show that games with such rules have nice properties. Specifically, while for general reward sharing an (integral) stable matching may not exist, for the reward sharing rules below we show they always exist (although only if there is no social context involved) and how to compute them efficiently. We also give improved bounds on prices of anarchy for these special cases. Specifically, we consider the following sharing rules:

- *Matthew Effect sharing:* In sociology, "Matthew Effect" is a term coined by Robert Merton to describe the phenomenon which says that, when doing similar work, the more famous person tends to get more credit than other less-known collaborators. We model such phenomena for our network by associating brand values λ_u with each node u, and defining the reward that node u gets by getting matched with node v as $r^u_{uv} = \frac{\lambda_u}{\lambda_u + \lambda_v} \cdot r_{uv}$. Thus nodes u and v split the edge reward in the ratio of $\lambda_u : \lambda_v$, and a node with high λ_u value gets a very large share of the reward.
- *Parasite sharing:* This effect is opposite to the Matthew effect in the sense that by collaborating with a renowned person, a less-known person becomes famous, whereas the reputation of the already renowned person does not change significantly from such a collaboration. We model this situation by defining the reward that node u gets by getting matched with node v as $r^u_{uv} = \frac{\lambda_v}{\lambda_u + \lambda_v} r_{uv}$. Thus nodes u and v split the edge reward in the ratio of $\lambda_v : \lambda_u$, in the exactly opposite way to the Matthew Effect sharing.
- *Additive sharing:* Often people collaborate based on not only the quality of a project but also how much the other person is comfortable to work with. We model such a situation by associating an intrinsic value β_u with each node u, which represents how pleasant they are to work with. Each edge $\{u, v\}$ also has an inherent quality h_{uv}. Then, the reward obtained by node u from being matched with node v is $r^u_{uv} = h_{uv} + \beta_v$.

For the sake of analysis, Matthew Effect sharing and Parasite sharing are the same if we change λ_u of Parasite sharing to $1/\lambda_u$ of Matthew Effect sharing. We will refer to both these models as Matthew Effect sharing from now on.

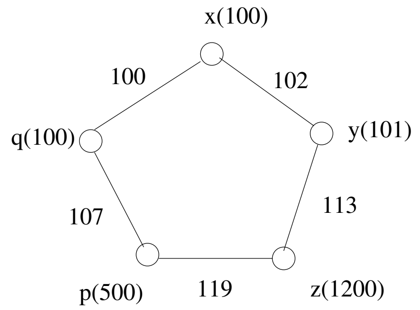

Fig. 8 Existence of a stable matching without friendship does not guarantee existence of a stable matching with friendship

Existence Recall that by Theorem 5 a fractional stable matching always exists for unequal reward sharing with friendship utilities. However, once we have friendship utilities, even the above intuitive special cases of reward sharing do not guarantee the existence of an *integral* stable matching. To see this, consider the following example.

Example 7. Consider the Matthew Effect sharing example as shown in Fig. 8. Edge labels indicate edge rewards, values in the brackets beside a node label are the brand values (λ values). By Theorem 8, for $\boldsymbol{\alpha} = \mathbf{0}$ a stable matching always exists for Matthew Effect sharing. Let us analyze the example in Fig. 8 with $\alpha_1 = 4/5, \alpha_2 = \alpha_3 = \cdots = 0$. We have

$$q^q_{qx} = 90 > q^q_{pq} = 89.1667$$
$$q^x_{xy} = 91.7493 > q^x_{qx} = 90$$
$$q^y_{yz} = 92.1545 > q^y_{xy} = 91.8507$$
$$q^z_{zp} = 112 > q^z_{yz} = 111.2455$$
$$q^p_{pq} = 103.4333 > q^p_{zp} = 102.2$$

Suppose there exists a stable (integral) matching. In such a matching exactly one node would stay unmatched. Consider a candidate matching $\{\{q, x\}, \{z, p\}\}$. Now y is unmatched and $\{x, y\}$ is a blocking pair, because $q^x_{xy} > q^x_{qx}$ and $q^y_{xy} > \alpha_1 r^x_{qx}$. Hence $\{\{q, x\}, \{z, p\}\}$ is not a stable matching. Similarly every other matching can be shown not to be stable. Thus, there exists no integral stable matching in this example with friendship utilities, even though without friendship (i.e., $\boldsymbol{\alpha} = \mathbf{0}$) an integral stable matching exists. ■

Example 7 shows that no stable (integral) matching may exist with friendship utilities. Without friendship, however, an integral stable matching exists and can be efficiently computed for Matthew Effect sharing and additive sharing, unlike in the case of general reward sharing [23].

Theorem 8. *An integral stable matching always exists in stable matching games with Matthew Effect sharing and additive sharing if* $\boldsymbol{\alpha} = 0$ *(i.e., if there is no friendship). Furthermore, this matching can be found in* $O(|V||E|)$ *time.*

Proof. Let us define a *preference cycle* as a cycle $(u_1, u_2, \cdots, u_k)$ in the graph G such that $r^{u_i}_{u_i u_{i+1}} \geq r^{u_i}_{u_i u_{i-1}}$ with at least one inequality being strict. Chung [19] defines *odd rings* and proves that if a graph does not contain odd rings, then a stable matching exists. It is straightforward to see that absence of preference cycles implies absence of odd rings. Hence, if a graph has no preference cycles, then a stable matching must exist. Below we prove the stronger statement that such a matching can also be found efficiently.

In brief, we show below that whenever there exist no preference cycles in a graph, we can always find two nodes which prefer getting matched to each other over other nodes. We allow them to get matched to each other and eliminate such matched nodes from the graph. Neither of these two nodes will ever deviate from this matching. Applying the same greedy scheme on the reduced graph will give us a stable matching. Then we will prove that this algorithm produces a stable matching in $O(|V||E|)$ time. Let us now proceed to the details.

Let T_u denote the sets of "best" neighbors of u as follows:

$$T_u = \{v \in N_1(u) : r^u_{uv} \geq r^u_{uw} \; \forall \{u, w\} \in E\} \ . \tag{23}$$

Now we construct a directed graph G_D as follows. For all nodes u, choose a node $v \in T_u$ and draw an edge from u directed to v. Every node in this graph has one outgoing edge, so this graph contains a (directed) cycle. If we find a cycle of length 2 then we have found two nodes which prefer each other the most. If a (directed) cycle $(u_1, u_2, \ldots, u_k)$ has length $k > 2$, then we have $r^{u_i}_{u_i u_{i+1}} \geq r^{u_i}_{u_i u_{i-1}}$. Now we cannot have $r^{u_2}_{u_2 u_3} > r^{u_2}_{u_1 u_2}$, otherwise in the original graph G, $(u_1, u_2, \ldots, u_k)$ would have constituted a preference cycle. Hence we have $r^{u_2}_{u_1 u_2} = r^{u_2}_{u_2 u_3}$. Thus u_1 and u_3 both are u_2's most preferred nodes. But we also have u_1 prefer u_2 the most as G_D has an edge from u_1 to u_2. Hence u_1 and u_2 is the pair of nodes that prefer each other the most.

Therefore we will always be able to find two nodes in G which prefer each other the most in their preference lists. Match them to each other and they will never have incentive to deviate from this matching. Remove these two nodes and repeat the procedure until no more nodes can be matched. Because no nodes matched in this process will ever deviate, we obtain a stable matching.

It takes $O(|E|)$ time to find each matched pair because for each edge we check if two nodes prefer each other the most. Since the total number of nodes to be matched are $O(|V|)$, we find a stable matching in $O(|V||E|)$ time, as long as there are no preference cycles. All that is left to show is that Matthew effect sharing and additive sharing do not lead to preference cycles.

Suppose a preference cycle exists in Matthew Effect sharing. Then there exists a cycle $(u_1, u_2, \ldots, u_k)$ such that

$$\frac{\lambda_{u_i}}{\lambda_{u_i} + \lambda_{u_{i+1}}} \cdot r_{u_i u_{i+1}} \geq \frac{\lambda_{u_i}}{\lambda_{u_i} + \lambda_{u_{i-1}}} \cdot r_{u_i u_{i-1}} \tag{24}$$

with at least one inequality being strict. Multiplying all these inequalities and canceling common factors, we reach a contradiction that $1 > 1$. Thus, a preference cycle cannot exist in Matthew Effect sharing.

Suppose a preference cycle exists in additive sharing. Then there exists a cycle $(u_1, u_2, \ldots, u_k)$ such that

$$h_{u_i u_{i+1}} + \beta_{u_{i+1}} \geq h_{u_i u_{i-1}} + \beta_{u_{i-1}} \tag{25}$$

with at least one inequality being strict. Adding all these inequalities and canceling common factors, we reach a contradiction that $0 > 0$. Thus, a preference cycle cannot exist in additive sharing. □

Price of Anarchy The price of anarchy of Matthew effect sharing can be as high as the guarantee of Theorem 6, with $R = \max_{\{u,v\}} \frac{\lambda_u}{\lambda_v}$. For additive sharing, however, things are much better. For a proof of this theorem see Appendix A.2.

Theorem 9. *The price of anarchy for (fractional) stable matching games with additive sharing and friendship utilities is at most* $\max\{2 + 2\alpha_1, 3\}$.

4 Future Directions and Contribution Games

We showed that the presence of a social context, such as friendship or altruism, can make a large difference in the existence and the quality of stable matchings, especially if the rewards obtained by neighboring nodes are unequal/unfair. Most of our results can be extended (with minor modifications) to contribution games [8] as well, as they can be considered non-standard fractional versions of stable matching. For details, see our arXiv preprint at [5].

There are many fascinating open problems resulting from our work. While we have established existence of (fractional) stable matchings for a quite large class of games, bounding the price of anarchy and stability in general, for other classes of reward sharing rules, or for other classes of influence values remains as a fascinating field for future study. In addition, structural results are of interest, e.g., concerning polyhedral properties of fractional and integral stable matchings, or the (non-)emptiness of the core in a stable matching game with externalities. Finally, there are interesting questions regarding the design and analysis of (polynomial-time computable) truthful mechanisms (with or without monetary transfers).

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A Appendix: Proofs on the Price of Anarchy

A.1 Proof of Theorem 6

We first introduce some notation. We denote by M^* an optimum fractional matching and use M to denote a fractional or integral stable matching. Let x^*_{uv} (or x_{uv}) denote the fraction of edge $\{u,v\}$ present in M^* (or M). Furthermore,

$$\begin{aligned}
S &= \{e \in E(G) : x^*_e > x_e\} \\
T &= \{e \in E(G) : x^*_e \leq x_e\} \\
S_u &= \{e \in E(G) : e \text{ is incident on } u \text{ and } e \in S\} \\
T_u &= \{e \in E(G) : e \text{ is incident on } u \text{ and } e \in T\} \\
E_u &= \text{All edges incident on } u\text{. Note that } E_u = S_u \cup T_u \\
y_u &= 1 - \sum_{\{u,v\} \in E_u} x_{uv} \\
y^*_u &= 1 - \sum_{\{u,v\} \in E_u} x^*_{uv} \\
\Delta_u &= \sum_{\{u,v\} \in T_u} (x_{uv} - x^*_{uv}) + \max\{(y_u - y^*_u), 0\}
\end{aligned}$$

The idea of the proof is the following. Consider an edge $\{u,v\} \in S$. If the fraction of the edge $\{u,v\}$ in M was increased to x^*_{uv} from x_{uv} by decreasing fractions of some other edges in M incident on u and v, then at least one of the endpoints of $\{u,v\}$ does not improve its utility. Tag this endpoint as corresponding to edge $\{u,v\}$ and denote the set of tagged nodes by B. We get one inequality for each node $u \in B$ for such a modification of fractions of adjoining edges. We will show that adding all such inequalities gives us the following:

$$\sum_{\{u,v\} \in S, u \in B} q^u_{uv} \cdot (x^*_{uv} - x_{uv}) \leq \sum_{u \in B} \sum_{\{u,w\} \in T_u} q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \ . \tag{26}$$

We prove (26) separately below. Now, recall that $Q = \max_{\{u,v\} \in E(G)} \frac{q^u_{uv}}{q^v_{uv}}$. Thus we have $q^u_{uv} \geq q_{uv}/(1+Q)$, giving us

$$\sum_{\{u,v\} \in S, u \in B} \frac{1}{1+Q} q_{uv} \cdot (x^*_{uv} - x_{uv}) \leq \sum_{u \in B} \sum_{\{u,w\} \in T_u} q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \ .$$

Now as discussed above, for every edge $\{u,v\} \in S$, we tag one of the endpoints (say u) and include it in the set B. Thus the left summation in the above equation can be simplified to obtain:

$$\sum_{\{u,v\} \in S} \frac{1}{1+Q} q_{uv} \cdot (x^*_{uv} - x_{uv}) \leq \sum_{u \in B} \sum_{\{u,w\} \in T_u} q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \ .$$

Using $T_u \subseteq T$ with $q_{uw} = q^u_{uw} + q^w_{uw}$, the above equation becomes

$$\sum_{\{u,v\} \in S} \frac{1}{1+Q} q_{uv} \cdot (x^*_{uv} - x_{uv}) \leq \sum_{\{u,w\} \in T} q_{uw} \cdot (x_{uw} - x^*_{uw}) \ .$$

Using algebraic simplifications and the fact that we have $1 \geq 1/(1+Q)$, we get the following sequence of inequalities:

$$\sum_{\{u,v\}\in S} \frac{1}{1+Q} \cdot q_{uv} \cdot x^*_{uv} + \sum_{\{u,v\}\in T} q_{uv} \cdot x^*_{uv}$$
$$\leq \sum_{\{u,w\}\in T} q_{uw} \cdot x_{uw} + \sum_{\{u,w\}\in S} \frac{1}{1+Q} \cdot q_{uw} \cdot x_{uw}$$
$$\Rightarrow \quad \sum_{\{u,v\}\in S\cup T} \frac{1}{1+Q} \cdot q_{uv} \cdot x^*_{uv} \leq \sum_{\{u,w\}\in S\cup T} q_{uw} \cdot x_{uw}$$
$$\Rightarrow \quad \frac{\sum_{\{u,v\}\in G} q_{uv} \cdot x^*_{uv}}{\sum_{\{u,w\}\in G} q_{uw} \cdot x_{uw}} \leq 1+Q \ , \tag{27}$$

where for the last inequality we have used the fact that $S \cup T$ covers all the edges in the graph. This proves the claim.

It remains to prove (26). Suppose node u gets tagged for edge $\{u,v\}$ if we increase x_{uv} to x^*_{uv} together with doing the following two compensatory steps:

- Decrease fraction x_{uw} of each $\{u,w\} \in T_u$ by $(x^*_{uv} - x_{uv})(x_{uw} - x^*_{uw})/\Delta_u$ and decrease fraction of each $\{v,z\} \in T_v$ by $(x^*_{uv} - x_{uv})(x_{vz} - x^*_{vz})/\Delta_v$ AND
- If $y_u > y^*_u$, then decrease y_u by $(x^*_{uv} - x_{uv})(y_u - y^*_u)/\Delta_u$, and if $y_v > y^*_v$, then decrease y_v by $(x^*_{uv} - x_{uv})(y_v - y^*_v)/\Delta_v$.

If the utility of u does not improve, then

$$q^u_{uv} \cdot (x^*_{uv} - x_{uv}) \leq \sum_{\{u,w\}\in T_u} q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \cdot c^{uv}_u$$
$$+ \sum_{\{v,z\}\in T_v} \alpha_1 r^v_{vz} \cdot (x_{vz} - x^*_{vz}) \cdot c^{uv}_v$$
$$+ \sum_{\{v,z\}\in T_v} \alpha_{uz} r^z_{vz} \cdot (x_{vz} - x^*_{vz}) \cdot c^{uv}_v \ , \tag{28}$$

where $c^{uv}_u = (x^*_{uv} - x_{uv})/\Delta_u$ and likewise for c^{uv}_v. To understand (28), note that $q^u_{uv} \cdot (x^*_{uv} - x_{uv})$ denotes the utility gained by u on edge $\{u,v\}$ by increasing x_{uv} to x^*_{uv}. The term $q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \cdot c^{uv}_u$ denotes the utility lost by u because of decreasing x_{uw} for an edge $\{u,w\} \in T_u$. When x_{vz} decreases for an edge $\{v,z\} \in T_v$ then by virtue of friendship with v, node u loses $\alpha_1 r^v_{vz} \cdot (x_{vz} - x^*_{vz}) \cdot c^{uv}_v$. When x_{vz} decreases for an edge $\{v,z\} \in T_v$ then depending on α_{uz} node u loses $\alpha_1 r^z_{vz} \cdot (x_{vz} - x^*_{vz}) \cdot c^{uv}_v$. Note that decreasing y_u is important when x_{uv} cannot be increased to x^*_{uv} without decreasing y_u.
As $\alpha_1 r^v_{vz} + \alpha_{uz} r^z_{vz} \leq r^v_{vz} + \alpha_1 r^z_{uz} = q^v_{vz}$, one can simplify (28) to

$$q^u_{uv} \cdot (x^*_{uv} - x_{uv}) \leq \sum_{\{u,w\}\in T_u} q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \cdot c^{uv}_u + \sum_{\{v,z\}\in T_v} q^v_{vz} \cdot (x_{vz} - x^*_{vz}) \cdot c^{uv}_v \tag{29}$$

We can form one such inequality for all edges $\{u,v\} \in S$. Let us inspect the coefficient of a term $q^u_{uw}(x_{uw} - x^*_{uw})$ appearing on the right hand side if we add all these inequalities. Notice that $q^u_{uw}(x_{uw} - x^*_{uw})c^{uv}_u$ appears only once for each edge $\{u,v\} \in S$ adjoining u. These are precisely the edges in S_u. Thus the coefficient of $q^u_{uw}(x_{uw} - x^*_{uw})$ will be $\sum_{\{u,v\}\in S_u} c^u_{uv}$ if we add all the inequalities formed on the lines of (29). By definition of c^u_{uv}, we have $\sum_{\{u,v\}\in S_u} c^u_{uv}$ is at most 1. Thus a term $q^u_{uw}(x_{uw} - x^*_{uw})$ can appear with coefficient at most 1 if we add all these inequalities formed on the lines of (29). Thus we get,

$$\sum_{\{u,v\}\in S, u\in B} q^u_{uv} \cdot (x^*_{uv} - x_{uv}) \leq \sum_{u\in B} \sum_{\{u,w\}\in T_u} q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \ .$$

We have proved (26) which in turn proves our claim.

The following example shows tightness of our bound.

Example 8. Consider the path as shown in Fig. 3. Set $\alpha_2 = \alpha_3 = \cdots = 0$ and use the following values:

$$r^u_{uv} = \frac{1}{1+\alpha_1} \qquad r^v_{uv} = \frac{1}{1+\alpha_1}$$
$$r^u_{uw} = \frac{1}{1+\alpha_1 R} \qquad r^w_{uw} = \frac{R}{1+\alpha_1 R}$$
$$r^v_{vz} = \frac{1}{1+\alpha_1 R} \qquad r^z_{vz} = \frac{R}{1+\alpha_1 R}$$

As desired, we have $\max_{\{x,y\}\in E(G)} \frac{r^x_{xy}}{r^y_{xy}} = R$. Using $q^x_{xy} = r^x_{xy} + \alpha r^y_{xy}$, we obtain

$$q^u_{uv} = 1 \qquad q^v_{uv} = 1$$
$$q^u_{uw} = 1 \qquad q^w_{uw} = Q$$
$$q^v_{vz} = 1 \qquad q^z_{vz} = Q$$

As desired, we have $\max_{\{x,y\}\in E(G)} \frac{q^x_{xy}}{q^y_{xy}} = \frac{R+\alpha_1}{1+\alpha_1 R} = Q$. We have $\{\{u,v\}\}$ as a stable matching: Given this matching, $\{u,w\}$ is not a blocking pair, because $q^u_{uw} \le q^u_{uv}$. Similarly, $\{v,z\}$ is not a blocking pair in matching $\{\{u,v\}\}$. Another stable matching is $\{\{u,w\},\{v,z\}\}$: Given this matching, $\{u,v\}$ will not be a blocking pair, becasue $q^u_{uv} < q^u_{uw} + \alpha_1 r^v_{vz}$, and so the condition in inequality (17) is violated. Since there are no other stable matchings for this graph, the price of anarchy will be determined by the value of the worst stable matching which is $\{\{u,v\}\}$. It is given by

$$\frac{r_{uw}+r_{vz}}{r_{uv}} = \frac{q_{uw}+q_{vz}}{q_{uv}} = 1+Q \ .$$

This completes the proof of Theorem 6. □

A.2 Proof of Theorem 9

Recall that in additive sharing, we associate a value β_u with each node u: how pleasant they are to work with. Each edge $\{u,v\}$ also has an inherent quality h_{uv}. Then, we model the edge reward obtained by node u from partnering with node v as $r^u_{uv} = h_{uv} + \beta_v$. Expressions for (friendship) utility of a node and the conditions for swivel or biswivel remain unchanged.

Before proceeding to the proof, we first introduce some notation. We denote by M^* an optimum fractional matching and by M a fractional or integral stable matching. Let x^*_{uv} (or x_{uv}) denote the fraction of edge $\{u,v\}$ present in M^* (or M). Furthermore,

$$\begin{aligned}
S &= \{e \in E(G) : x^*_e > x_e\}\\
T &= \{e \in E(G) : x^*_e \le x_e\}\\
S_u &= \{e \in E(G) : e \text{ is incident on } u \text{ and } e \in S\}\\
T_u &= \{e \in E(G) : e \text{ is incident on } u \text{ and } e \in T\}\\
E_u &= \text{All edges incident on } u. \text{ Note that } E_u = S_u \cup T_u\\
y_u &= 1 - \sum_{\{u,v\}\in E_u} x_{uv}\\
y^*_u &= 1 - \sum_{\{u,v\}\in E_u} x^*_{uv}\\
\Delta_u &= \sum_{\{u,v\}\in T_u} (x_{uv} - x^*_{uv}) + \max\{(y_u - y^*_u), 0\}\\
c^{uv}_u &= (x^*_{uv} - x_{uv})/\Delta_u
\end{aligned}$$

We will show later in the proof that the following holds

$$\sum_{\{u,v\}\in S}\left(h_{uv}+\frac{\beta_u}{2}\;+\;\frac{\beta_v}{2}\right)(x^*_{uv}-x_{uv})$$
$$\leq \max\{2+2\alpha_1,3\}\cdot\sum_{\{u,w\}\in T}\left(h_{uw}+\frac{\beta_u}{2}+\frac{\beta_w}{2}\right)(x_{uw}-x^*_{uw}) \quad (30)$$

With some simple algebraic manipulations, we get

$$\sum_{\{u,v\}\in S}(2h_{uv}+\beta_u+\beta_v)\cdot x^*_{uv}+\sum_{\{u,v\}\in T}\max\{2+2\alpha_1,3\}\cdot(2h_{uv}\;+\beta_u+\beta_v)\cdot x^*_{uv}$$
$$\leq \sum_{\{u,w\}\in T}\max\{2+2\alpha_1,3\}\cdot(2h_{uw}+\beta_u+\beta_w)\cdot x_{uw}$$
$$+\sum_{\{u,w\}\in S}(2h_{uw}+\beta_u+\beta_w)\cdot x_{uw}$$

Using $\max\{2+2\alpha_1,3\}>1$, the above equation leads us to our result

$$\sum_{\{u,v\}\in S\cup T}(2h_{uv}\;+\beta_u+\beta_v)\cdot x^*_{uv}$$
$$\leq \sum_{\{u,w\}\in S\cup T}\max\{2+2\alpha_1,3\}\cdot(2h_{uw}+\beta_u+\beta_w)\cdot x_{uw}\;. \quad (31)$$

Now $S\cup T$ covers all the edges in the graph. Also, for additive sharing with friendship utilties we have $q^u_{uv}=(1+\alpha_1)h_{uv}+\alpha_1\beta_u+\beta_v$, thus $q_{xy}=q^x_{xy}+q^y_{xy}=(1+\alpha_1)(2h_{uv}+\beta_u+\beta_v)$. Using this, Eqn (31) implies that the price of anarchy is at most $\max\{2+2\alpha_1,3\}$. All that remains to show is Eqn (30) which we do next.

To start with, notice that each edge $\{u,v\}\in S$ can be classified into two categories: a) $\{u,v\}\in S$ such that $y_u\leq y^*_u$ and $y_v\leq y^*_v$ AND b) $\{u,v\}\in S$ such that $y_u>y^*_u$ and $y_v\leq y^*_v$. Note that we cannot have both $y_u>y^*_u$ and $y_v>y^*_v$, because then the fraction x_{uv} can be increased by $\min(y_u-y^*_u,y_v-y^*_v)$ by decreasing y_u and y_v by the same quantity. Both u and v would improve their utility in such a case, thus M could not be a (fractional) stable matching.

We will show that whichever category $\{u,v\}\in S$ belongs to, for one of the endpoints, say u, the following inequality will hold true with $\zeta=\max\{1/2,\alpha_1\}$:

$$\left(h_{uv}+\frac{1}{2}\beta_u+\frac{1}{2}\beta_v\right)(x^*_{uv}\;-x_{uv})$$
$$\leq \sum_{\{u,w\}\in T_u}\left((1+\alpha_1)h_{uw}+\frac{1}{2}\beta_u+\beta_w\right)(x_{uw}-x^*_{uw})c^{uv}_u$$
$$+\sum_{\{v,z\}\in T_v}\left((1+\alpha_1)h_{vz}+\zeta\beta_v+\zeta\beta_z\right)(x_{vz}-x^*_{vz})c^{uv}_v \quad (32)$$

We call u a *witness* node for $\{u,v\}\in S$. We will observe that adding all the inequalities like Eqn (32) corresponding to each edge $\{u,v\}\in S$ leads us to Eqn (30), thus proving the theorem in turn. Now let us see how Eqn (32) can be proved for every edge $\{u,v\}\in S$. As mentioned before, $\{u,v\}\in S$ can be classified into two categories and we will prove Eqn (32) for each of them.

1. $\{u,v\}\in S$ **with** $y_u\leq y^*_u$ **and** $y_v\leq y^*_v$**:** Here we increase x_{uv} to x^*_{uv} by decreasing fraction of each $\{u,w\}\in T_u$ by $(x_{uw}-x^*_{uw})c^u_{uv}$ and decreasing fraction of each $\{v,z\}\in T_v$ by $(x_{vz}-x^*_{vz})c^v_{uv}$. As M is a stable matching, this does not improve the utility of at least one of the endpoints of $\{u, v\}$, say u. Call u a witness node for edge $\{u, v\}$. Since the utility of node u does not improve, we get

$$q_{uv}(x^*_{uv} - x_{uv}) \le \sum_{\{u,w\}\in T_u} q_{uw}(x_{uw} - x^*_{uw})c^{uv}_u + \alpha_1 \cdot \sum_{\{v,z\}\in T_v} r^v_{vz}(x_{vz} - x^*_{vz})c^{uv}_v \\ + \alpha_{uz} \cdot \sum_{\{v,z\}\in T_v} r^z_{vz}(x_{vz} - x^*_{vz})c^{uv}_v \ . \tag{33}$$

Eqn (33) can be explained as follows: Its left-hand side represents utility gained by u by increasing x_{uv} to x^*_{uv}. The first summation on the right hand side represents the utility lost by u because of decreasing x_{uw} by $(x_{uw} - x^*_{uw})c^{uv}_u$ for each $\{u, w\} \in T_u$. The second summation on the right hand side represents the utility lost by u by virtue of friendship with v when we decrease x_{vz} by $(x_{vz} - x^*_{vz})c^{uv}_v$ for each $\{v, z\} \in T_v$. The third summation on the right hand side represents the utility lost by u by virtue of friendship with w when we decrease x_{vz} by $(x_{vz} - x^*_{vz})c^{uv}_v$ for each $\{v, z\} \in T_v$. Because for additive sharing we have $r^x_{xy} = h_{xy} + \beta_u$ and $q^x_{xy} = (1 + \alpha_1)h_{uv} + \alpha_1\beta_u + \beta_v$, Eqn (33) implies that

$$((1 + \alpha_1)h_{uv} + \alpha_1\beta_u + \beta_v)(x^*_{uv} - x_{uv}) \\ \le \sum_{\{u,w\}\in T_u} ((1 + \alpha_1)h_{uw} + \alpha_1\beta_u + \beta_w)(x_{uw} - x^*_{uw})c^{uv}_u \\ + \sum_{\{v,z\}\in T_v} ((\alpha_1 + \alpha_{uz})h_{vz} + \alpha_{uz}\beta_v + \alpha_1\beta_z)(x_{vz} - x^*_{vz})c^{uv}_v \ . \tag{34}$$

Eqn (34) has similar interpretation as Eqn (33). To simplify Eqn (34), notice that when $y_u \le y^*_u$, calculating Δ_u does not involve $y_u - y^*_u$ giving us $\sum_{\{u,w\}\in T_u}(x_{uw} - x^*_{uw})c^{uv}_u = (x^*_{uv} - x_{uv})$. Thus we have

$$(1/2 - \alpha_1)\beta_u(x^*_{uv} - x_{uv}) = (1/2 - \alpha_1)\beta_u \sum_{\{u,w\}\in T_u} (x_{uw} - x^*_{uw})c^{uv}_u \ . \tag{35}$$

Let us add Eqn (35) to Eqn (34) and replace β_v by $\beta_v/2$ on the left-hand side. Additionally, using $\alpha_{uz} \le \alpha_1 \le \zeta$ results into the following equation:

$$\left((1 + \alpha_1)h_{uv} + \frac{1}{2}\beta_u + \frac{1}{2}\beta_v\right)(x^*_{uv} - x_{uv}) \\ \le \sum_{\{u,w\}\in T_u} \left((1 + \alpha_1)h_{uw} + \frac{1}{2}\beta_u + \beta_w\right)(x_{uw} - x^*_{uw})c^{uv}_u \\ + \sum_{\{v,z\}\in T_v} ((1 + \alpha_1)h_{vz} + \zeta\beta_v + \zeta\beta_z)(x_{vz} - x^*_{vz})c^{uv}_v \ . \tag{36}$$

Thus we have proved (32) holds for a node u acting as witness for $\{u, v\} \in S$ such that $y_u \le y^*_u$ and $y_v \le y^*_v$.

2. **$\{u, v\} \in S$ with $y_u > y^*_u$ and $y_v \le y^*_v$:** Here we find a constant $\epsilon > 0$ such that $\epsilon \cdot (x^*_{uv} - x_{uv}) \le y_u - y^*_u$. Then we increase x_{uv} by $\epsilon \cdot (x^*_{uv} - x_{uv})$ by decreasing y_u by the same amount and by decreasing each $\{v, z\} \in T_v$ by $\epsilon \cdot (x_{vz} - x^*_{vz})c^{uv}_v$. By doing this the utility of at least one of the endpoints of $\{u, v\}$, does not improve because M is a stable matching. This endpoint can be either u or v. We will prove that Eqn (32) holds for each of these cases.

2(a): Suppose the utility of node u does not improve: Call u a witness node for edge $\{u,v\} \in S$. Since the utility of node u does not improve, we have

$$\begin{aligned}((1+\alpha_1)h_{uv} &+ \alpha_1\beta_u + \beta_v)\cdot\epsilon\cdot(x^*_{uv} - x_{uv}) \\ &\le \sum_{\{v,z\}\in T_v} ((\alpha_1+\alpha_{uz})h_{vz} + \alpha_{uz}\beta_v + \alpha_1\beta_z)\cdot\epsilon(x_{vz} - x^*_{vz})c_v^{uv}\end{aligned} \tag{37}$$

Eqn (37) can be explained on the similar lines of Eqn (33) and (34). The only difference being that the first summation from Eqn (34) is absent from Eqn (37) as fractions x_{uw} do not change for $\{u,w\} \in T_u$. Canceling ϵ from each side of Eqn (37), we get

$$((1+\alpha_1)h_{uv} + \alpha_1\beta_u + \beta_v)\cdot(x^*_{uv} - x_{uv}) \le \sum_{\{v,z\}\in T_v} ((\alpha_1+\alpha_{uz})h_{vz} + \alpha_{uz}\beta_v + \alpha_1\beta_z)\cdot(x_{vz} - x^*_{vz})c_v^{uv} \tag{38}$$

$$\Rightarrow \alpha_1\beta_u(x^*_{uv} - x_{uv}) \le \sum_{\{v,z\}\in T_v} ((\alpha_1+\alpha_{uz})h_{vz} + \alpha_{uz}\beta_v + \alpha_1\beta_z)\cdot(x_{vz} - x^*_{vz})c_v^{uv} \tag{39}$$

We multiply Eqn (39) by $(1/2 - \alpha_1)/\alpha_1$ on both sides and add it to Eqn (38) to get the following:

$$\Big((1+\alpha_1)h_{uv} + \frac{\beta_u}{2} + \beta_v)\cdot(x^*_{uv} - x_{uv}) \le \sum_{\{v,z\}\in T_v} \left(\frac{\alpha_1+\alpha_{uz}}{2\alpha_1}h_{vz} + \frac{\alpha_{uz}}{2\alpha_1}\beta_v + \frac{1}{2}\beta_z\right)\cdot(x_{vz} - x^*_{vz})c_v^{uv} \tag{40}$$

Note that to get to Eqn (40), we are performing division by α_1 which requires $\alpha_1 > 0$. However, also notice that if $\alpha_1 = 0$ (and hence all $\alpha_i = 0$) and $y_u > y^*_u$, then node u can only improve its utility by increasing x_{uv} by $\epsilon\cdot(x^*_{uv} - x_{uv})$. Thus the case of $y_u > y^*_u$ and u not improving its utility does not arise.

Now replacing β_v by $\beta_v/2$ and using $\alpha_{uz} \le \alpha_1$ in Eqn (40), we get

$$\begin{aligned}
&\left((1+\alpha_1)h_{uv} + \frac{\beta_u}{2} + \frac{\beta_v}{2}\right)\cdot(x^*_{uv} - x_{uv}) \\
&\qquad\le \sum_{\{v,z\}\in T_v}\left(h_{vz} + \frac{1}{2}\beta_v + \frac{1}{2}\beta_z\right)\cdot(x_{vz} - x^*_{vz})c_v^{uv} \\
\Rightarrow\ &\left((1+\alpha_1)h_{uv} + \frac{\beta_u}{2} + \frac{\beta_v}{2}\right)\cdot(x^*_{uv} - x_{uv}) \\
&\qquad\le \sum_{\{u,w\}\in T_u}\left((1+\alpha_1)h_{uw} + \frac{1}{2}\beta_u + \beta_w\right)(x_{uw} - x^*_{uw})c_u^{uv} \\
&\qquad\quad + \sum_{\{v,z\}\in T_v}\left(h_{vz} + \frac{1}{2}\beta_v + \frac{1}{2}\beta_z\right)\cdot(x_{vz} - x^*_{vz})c_v^{uv} \\
\Rightarrow\ &\left((1+\alpha_1)h_{uv} + \frac{1}{2}\beta_u + \frac{1}{2}\beta_v\right)(x^*_{uv} - x_{uv}) \\
&\qquad\le \sum_{\{u,w\}\in T_u}\left((1+\alpha_1)h_{uw} + \frac{1}{2}\beta_u + \beta_w\right)(x_{uw} - x^*_{uw})c_u^{uv} \\
&\qquad\quad + \sum_{\{v,z\}\in T_v}((1+\alpha_1)h_{vz} + \zeta\beta_v + \zeta\beta_z)(x_{vz} - x^*_{vz})c_v^{uv}\ .
\end{aligned} \tag{41}$$

Thus, Eqn (32) holds in this case too.

2(b): Suppose the utility of node v does not improve. Call node v a witness node for edge $\{u, v\}$. As the utility of node v does not improve, we get

$$((1+\alpha_1)h_{uv} + \alpha_1\beta_v + \beta_u) \cdot \epsilon(x^*_{uv} - x_{uv}) \leq \sum_{\{v,z\} \in T_v} ((1+\alpha_1)h_{vz} + \alpha_1\beta_v + \beta_z) \cdot \epsilon(x_{vz} - x^*_{vz}) c^{uv}_v \ . \quad (42)$$

Eqn (42) can be explained similarly as Eqn (34) and (33). The only differences are that the roles of u and v are reversed and the summation corresponding to the utility lost by v because of decreasing x_{uw} is absent from the right hand side as fractions x_{uw} do not change for $\{u, w\} \in T_u$ in this case.
To simplify Eqn (42), we cancel ϵ from each side. We also note that as $y_v \leq y^*_v$, calculating Δ_v does not involve $y_v - y^*_v$ giving us $\sum_{\{v,z\} \in T_v} (x_{vz} - x^*_{vz}) c^{uv}_v = x^*_{uv} - x_{uv}$. This implies that

$$(1/2 - \alpha_1)\beta_v(x^*_{uv} - x_{uv}) = (1/2 - \alpha_1)\beta_v \sum_{\{v,z\} \in T_v} (x_{vz} - x^*_{vz}) c^{uv}_v \ . \quad (43)$$

Adding Eqn (43) to Eqn (42) and replacing β_u by $\beta_u/2$ on the left hand side, we get

$$\begin{aligned} &\left((1+\alpha_1)h_{uv} + \frac{1}{2}\beta_v + \frac{1}{2}\beta_u\right)(x^*_{uv} - x_{uv}) \\ &\qquad \leq \sum_{\{v,z\} \in T_v} \left((1+\alpha_1)h_{vz} + \frac{1}{2}\beta_v + \beta_z\right)(x_{vz} - x^*_{vz}) c^{uv}_v \\ \Rightarrow \ &\left((1+\alpha_1)h_{uv} + \frac{1}{2}\beta_v + \frac{1}{2}\beta_u\right)(x^*_{uv} - x_{uv}) \\ &\qquad \leq \sum_{\{v,z\} \in T_v} \left((1+\alpha_1)h_{vz} + \frac{1}{2}\beta_v + \beta_z\right)(x_{vz} - x^*_{vz}) c^{uv}_v \\ &\qquad\quad + \sum_{\{u,w\} \in T_u} ((1+\alpha_1)h_{uw} + \zeta\beta_w + \zeta\beta_w)(x_{uw} - x^*_{uw}) c^{uv}_u \ . \end{aligned} \quad (44)$$

Thus, Eqn (32) holds in this case as well.

We showed in cases 2(a) and 2(b) that for every edge $\{u, v\} \in S$, for one of its endpoints, say u, the inequality given by Eqn (32) holds true. Now we will show that adding these inequalities leads us to Eqn (30) which in turn proves the theorem as discussed before. Let us look at the coefficients of various terms after we add these inequalities:

- A term $h_{uw}(x_{uw} - x^*_{uw})$ on the right hand side will have coefficient at most

$$(1+\alpha_1) \cdot (\sum_{\{u,v\} \in S} c^{uv}_u + \sum_{(w,x) \in S} c^{wx}_w) \ ,$$

  which is at most $2(1+\alpha_1)$ using $\sum_{\{x,y\} \in S} c^{xy}_x \leq 1$.
- Let $u = t(u, v)$ denote that u acts as witness node for edge $\{u, v\} \in S$. Then a term $\beta_u(x_{uw} - x^*_{uw})$ on the right hand side will appear with coefficient

$$\sum_{\substack{\{u,v\} \in S \\ u = t(u,v)}} \frac{1}{2} \cdot c^{uv}_u + \sum_{\substack{\{u,v\} \in S \\ u \neq t(u,v)}} \zeta \cdot c^{uv}_u + \sum_{\substack{\{w,x\} \in S \\ w = t(w,x)}} c^{wx}_w + \sum_{\substack{\{w,x\} \in S \\ w \neq t(w,x)}} \zeta \cdot c^{wx}_w \ .$$

  When $\alpha_1 \geq 1/2$, we have $\zeta = \alpha_1$, thus the coefficient of $\beta_u(x_{uw} - x^*_{uw})$ becomes

$$\sum_{\substack{\{u,v\} \in S \\ u = t(u,v)}} \frac{1}{2} \cdot c^{uv}_u + \sum_{\substack{\{u,v\} \in S \\ u \neq t(u,v)}} \alpha_1 \cdot c^{uv}_u + \sum_{\substack{\{w,x\} \in S \\ w = t(w,x)}} c^{wx}_w + \sum_{\substack{\{w,x\} \in S \\ w \neq t(w,x)}} \alpha_1 \cdot c^{wx}_w \ .$$

This quantity can be at most $\alpha_1 + 1$.
On the other hand, when $\alpha_1 < 1/2$, we have $\zeta = 1/2$, thus the coefficient of $\beta_u(x_{uw} - x^*_{uw})$ becomes

$$\sum_{\substack{\{u,v\}\in S \\ u=t(u,v)}} \frac{1}{2} \cdot c_u^{uv} + \sum_{\substack{\{u,v\}\in S \\ u\neq t(u,v)}} \frac{1}{2} \cdot c_u^{uv} + \sum_{\substack{\{w,x\}\in S \\ w=t(w,x)}} c_w^{wx} + \sum_{\substack{\{w,x\}\in S \\ w\neq t(w,x)}} \frac{1}{2} \cdot c_w^{wx} \ .$$

This quantity can be at most $3/2$.

Taking into account these coefficients when we sum the inequalities given by Eqn (32) over all $\{u, v\} \in S$, we get

$$\sum_{\{u,v\}\in S} \left(h_{uv} + \frac{\beta_u}{2} \ + \ \frac{\beta_v}{2}\right)(x^*_{uv} - x_{uv})$$
$$\leq \max\{2 + 2\alpha_1, 3\} \cdot \sum_{\{u,w\}\in T} \left(h_{uw} + \frac{\beta_u}{2} + \frac{\beta_w}{2}\right)(x_{uw} - x^*_{uw})$$

and thereby prove Eqn (30) and our claim in turn. □