Friendship and Stable Matching*

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Abstract

We study stable matching problems in networks where players are embedded in a social context, and may incorporate friendship relations or altruism into their decisions. Each player is a node in a social network and strives to form a good match with a neighboring player. We consider the existence, computation, and inefficiency of stable matchings from which no pair of players wants to deviate. When the benefits from a match are the same for both players, we show that incorporating the well-being of other players into their matching decisions significantly decreases the price of stability, while the price of anarchy remains unaffected. Furthermore, a good stable matching achieving the price of stability bound always exists and can be reached in polynomial time. We extend these results to more general matching rewards, when players matched to each other may receive different utilities from the match. For this more general case, we show that incorporating social context (i.e., "caring about your friends") can make an even larger difference, and greatly reduce the price of anarchy. We show a variety of existence results, and present upper and lower bounds on the prices of anarchy and stability for various matching utility structures.

1 Introduction

Stable matching problems capture the essence of many important assignment and allocation tasks in economics and computer science. The central approach to analyzing such scenarios is two-sided matching, which has been studied intensively since the 1970s in both the algorithms and economics literature. An important variant of stable matching is matching with cardinal utilities, when each match can be given numerical values expressing the *quality* or *reward* that the match yields for each of the incident players [6]. Cardinal utilities specify the quality of each match instead of just a preference ordering, and they allow the comparison of different matchings using measures such as social welfare. A particularly appealing special case of cardinal utilities is known as correlated stable matching, where both players who are matched together obtain the same reward. Apart from the wide-spread applications of correlated stable matching in, e.g., market sharing [20], job markets [9], social networks [22], and distributed computer networks [33], this model also has favorable theoretical properties such as the existence of a potential function. It guarantees existence of a stable matching even in the non-bipartite case, where every pair of players is allowed to match [3, 33].

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When matching individuals in a social environment, it is often unreasonable to assume that each player cares only about their own match quality. Instead, players incorporate the well-being of their friends/neighbors as well, or that of friends-of-friends. Players may even be altruistic to some degree, and consider the welfare of all players in the network. Caring about friends and altruistic behavior is commonly observed in practice and has been documented in laboratory experiments [17, 31]. In addition, in economics there exist recent approaches towards modelling and analyzing *other-regarding preferences* [18]. Given that other-regarding preferences are widely observed in practice, it is an important fundamental question to model and characterize their influence in classic game-theoretical environments. Very recently, the impact of social influence on congestion and potential games has been characterized prominently in [11, 13–15, 23–25].

In this paper, we consider a natural approach to incorporate social effects into partner selection and matching scenarios. In particular, we study how social context influences stability and efficiency in matching games. Our model to incorporate social context into player decisions is similar to recent approaches in algorithmic game theory and uses dyadic influence values tied to the hop distance in the graph. In this way, every player may consider the well-being of every other player to some degree, with the degree of this regardfulness possibly decaying with the hop distance. The perceived utility of a player is then composed of a weighted average of player utilities. Players who only care about their neighbors, as well as fully altruistic players, are special cases of this model.

Moreover, for matching in social environments, the standard model of correlated stable matching may be too constraining compared to general cardinal utilities, because matched players receive exactly the same reward. Such an *equal sharing* property is intuitive and bears a simple beauty, but there are a variety of other reward sharing methods that can be more natural in different contexts. For instance, in theoretical computer science it is common practice to list authors alphabetically, but in other disciplines the author sequence is carefully designed to ensure a proper allocation of credit to the different participants of a joint paper. Here the credit is often supposed to be allocated in terms of input, i.e., the first author should be the one that has contributed most to the project. Such input-based or proportional sharing is then sometimes overruled with sharing based on intrinsic or acquired social status, e.g., when a distinguished expert in a field is easily recognized and subconsciously credited most with authorship of an article. In this paper, we are interested in how such unequal reward sharing rules affect stable matching scenarios. In particular, we consider a large class of local reward sharing rules and characterize the impact of unequal sharing on existence and inefficiency of stable matchings, both in cases when players are embedded in a social context and when they are not.

1.1 Stable Matching Within a Social Context

Correlated stable matching is a prominent subclass of general ordinary stable matching. In this game, we are given a (non-bipartite) graph $G = (V, E)$ with edge weights r_e. In a matching M, if node u is matched to node v, the utility of node u is defined to be exactly r_e. This can be interpreted as both u and v getting an identical reward from being matched together. We will also consider unequal reward sharing, where node u obtains some reward r_e^u and node v obtains reward r_e^v with $r_e^u + r_e^v = r_e$. Therefore, the preference ordering of each node over its possible matches is implied by the rewards that this node obtains from different edges. A pair of nodes (u, v) is called a *blocking pair* in matching M if u and v are not matched to each other in M, but can both strictly increase their rewards by being matched to each other instead. A matching with no blocking pairs is called a *stable matching.*

While the matching model above has been well-studied, in this paper we are interested in stable matchings that arise in the presence of social context. Denote the reward obtained by a node v in a matching M as R_v. We now consider the case when node u not only cares about its own reward, but also about the rewards of its friends. Specifically, the *perceived* or *friendship utility* of node v in matching M is defined as

$$U_v = R_v + \sum_{d=1}^{diam(G)} \alpha_d \sum_{u \in N_d(v)} R_u,$$

where $N_d(v)$ is the set of nodes with shortest distance exactly d from v, and $1 \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq 0$ (we use $\vec{\alpha}$ to denote the vector of α_i values). In other words, for a node u that is distance d away from v, the utility of v increases by an α_d factor of the reward received by u. Thus, if $\alpha_d = 0$ for all $d \geq 2$, this means that nodes only care about their neighbors, while if all $\alpha_d > 0$, this means that nodes are altruistic and care about the rewards of everyone in the graph. The perceived utility is the quantity that the nodes are trying to maximize, and thus, in the presence of friendship, a blocking pair is a pair of nodes such that each node can increase its *perceived utility* by matching to each other. Given this defintion of blocking pair, a stable matching is again a matching without such a blocking pair.

We also study more general stable matching models with nodes u and v receiving different rewards r^u_{uv} and r^v_{uv} from an edge $(u, v) \in M$. Under these conditions, a stable matching is not guaranteed to exist. Instead, we resort to *fractional stable matchings* defined as follows. In a fractional matching M there is a real number $x_e \in [0, 1]$ for each edge e. It represents the degree to which edge e is "in the matching" and can be thought of as the strength of the match between the endpoints of e. In addition, for every node u there is a budget constraint $\sum_{e \ni u} x_e \leq 1$. Fractional matching is especially well-motivated in a social context, since it captures the idea of relationships of varying strengths. The budget constraint models the fact that a single person cannot be involved in an unlimited amount of strong relationships. With fractional link strengths, the reward of a node u for an edge e becomes $x_e \cdot r^u_e$.

A fractional stable matching is a fractional matching without blocking pairs. Analogous to a blocking pair for an integral matching, a blocking pair for a fractional matching is an edge (u, v) such that by increasing the strength of edge (u, v) (and possibly decreasing the strengths of some other edges (u, w) and (v, z) to keep the budget constraints), both u and v strictly improve their utilities. For fractional matching, the extension to friendships, social context, and perceived utility is straightforward. Througout the paper, the term *stable matching* refers to an integral stable matching. We will explicitly mention when fractional stable matchings are studied.

Centralized Optimum and the Price of Anarchy We study the social welfare of equilibrium solutions and compare them to an optimal centralized solution. The social welfare is the sum of rewards, i.e., a *social optimum* is a matching that maximizes $\sum_v R_v$. Notice that, while this is equivalent to maximizing the sum of player utilities when $\vec{\alpha} = 0$, this is no longer true with social context (i.e., when $\vec{\alpha} \neq 0$). Nevertheless, as in e.g. [14, 34], we believe this is a well-motivated and important measure of solution quality, as it captures the overall performance of the system, while ignoring the perceived "good-will" effects of friendship and altruism. For example, when considering projects done in pairs, the reward of an edge can represent actual productivity, while the perceived utility may not.

To compare stable solutions with a social optimum, we will often consider the price of anarchy and the price of stability. When considering stable matchings, by the price of anarchy (resp. stability) we will mean the ratio of social welfare of the social optimum and the social welfare of the worst (resp. best) stable matching.

1.2 Our Results

In Section 2, we consider stable matching with friendship utilities and equal reward sharing. In this case, a stable matching exists and the price of anarchy (ratio of the maximum-weight matching with the worst stable matching) is at most 2, the same as in the case without friendship. The price of stability, on the other hand, improves in the presence of friendship, as we can show a tight bound of $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$. Moreover, we present a dynamic process that converges to a stable matching of at least this quality in polynomial time, if initiated from the maximum-weight matching. Our results imply that for socially aware players, the price of stability can greatly improve: e.g., if $\alpha_1 = \alpha_2 = \frac{1}{2}$, then the price of stability is at most $\frac{6}{5}$, and a solution of this quality can be obtained efficiently.

In Section 3 we instead study general reward sharing schemes. When two nodes matched together may receive different rewards, an integral stable matching may not exist. Thus, we focus on *fractional* stable matchings which we show to always exist, even with friendship utilities. We show that for arbitrary reward sharing, prices of anarchy and stability depend on the level of inequality among reward shares. Specifically, if R is the maximum ratio over all edges $(u, v) \in E$ of the reward shares of node u and v, then the price of anarchy is at most $\frac{(1+R)(1+\alpha_1)}{1+\alpha_1 R}$. Thus, compared to the equal reward sharing case, if sharing is extremely unfair (R is unbounded), then friendship becomes even more important: changing α_1 from 0 to $\frac{1}{2}$ reduces the price of anarchy from unbounded to at most 3. In addition, for several particularly natural local reward sharing rules, we show that an integral stable matching exists, give improved price of anarchy guarantees, and show tight lower bounds.

1.3 Related Work

Stable matching problems have been studied intensively over the last few decades. On the algorithmic side, existence, efficient algorithms, and improvement dynamics for two-sided stable matchings have been of interest (for references, see standard textbooks [21, 36]). In this paper we address the more general stable roommates problem, in which every player can be matched to every other player. For general preference lists, there have been numerous works characterizing and algorithmically deciding existence of stable matchings [16, 26, 36, 37]. In contrast, fractional stable matchings are always guaranteed to exist and exhibit various interesting polyhedral properties [1, 2, 37]. For the correlated stable roommates problem, existence of (integral) stable matchings is guaranteed by a potential function argument [3, 33], and convergence time of random improvement dynamics is polynomial [4]. In [7], price of anarchy and stability bounds for *approximate* correlated stable matchings were provided. Similar studies in a setting with geometric distances were conducted in [10]. In contrast, we study friendship, altruism, and unequal reward sharing in stable roommates problems with cardinal utilities.

Another line of research closely connected to some of our results involves game-theoretic models for contribution. In [8] we consider a contribution game tied closely to matching problems. Here players have a budget of effort and contribute parts of this effort towards specific projects and relationships. For more related work on the contribution game, see [8]. All previous results for this model concern equal sharing and do not address the impact of the player's social context. As we discuss in the conclusion, many of our results for friendship utilities can also be extended to such contribution games.

Analytical aspects of reward sharing have been a central theme in game theory since its beginning, especially in cooperative games [35]. Recently, there have been prominent algorithmic results also for network bargaining [27,29] and credit allocation problems [28]. A recent line of work [38,39] treats extensions of cooperative games, where players invest into different coalitional projects. The

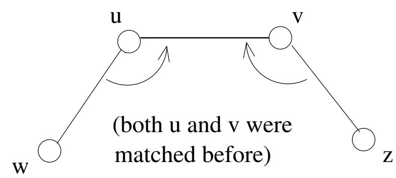

Figure 1: biswivel deviation

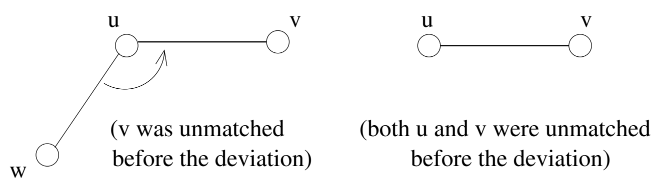

Figure 2: swivel deviation

main focus of this work is global design of reward sharing schemes to guarantee cooperative stability criteria. Our focus here is closer to, e.g., recent work on profit sharing games [12, 32]. We are interested in existence, computational complexity, and inefficiency of stable states under different reward sharing rules, with an aim to examine the impact of social context on stable matchings.

Our notion of a player's social context is based on numerical influence parameters that determine the impact of player rewards on the (perceived) utilities of other players. A recently popular model of altruism is inspired by Ledyard [30] and has generated much interest in algorithmic game theory [14, 15, 24]. In this model, each player optimizes a perceived utility that is a weighted linear combination of his own utility and the utilitarian welfare function. Similarly, for surplus collaboration [11] perceived utility of a player consists of the sum of players utilities in his neighborhood within a social network. Our model is similar to [13, 25] and smoothly interpolates between these global and local approaches.

2 Stable Matching with Equal Reward Sharing

We begin by considering correlated stable matching in the presence of friendship utilities. In this section, the reward received by both nodes of an edge in a matching is the same, i.e., we use equal reward sharing, where every edge e has an inherent value r_e and both endpoints receive this value if edge e is in the matching. We consider more general reward sharing schemes in Section 3. Recall that the friendship utility of a node v increases by $\alpha_d R_u$ for every node u, where d is the shortest distance between v and u. We abuse notation slightly, and let α_{uv} denote α_d, so if u and v are neighbors, then $\alpha_{uv} = \alpha_1$.

Given a matching M, let us classify the following types of improving deviations that a blocking pair can undergo.

Definition 1. *We call an improving deviation a* **biswivel** *whenever two neighbors u and v switch to match to each other, such that both u and v were matched to some other nodes before the deviation in M.*

See Figure 1 for explanation. For such a biswivel to exist in a matching, the following necessary and sufficient conditions must hold.

$$(1+\alpha_1)r_{uv} \quad > \quad (1+\alpha_1)r_{uw} + (\alpha_1+\alpha_{uz})\, r_{vz} \tag{1}$$

$$(1+\alpha_1)r_{uv} \quad > \quad (1+\alpha_1)r_{vz} + (\alpha_1+\alpha_{vw})\, r_{uw} \tag{2}$$

Intuitively, the left side of Inequality (1) quantifies the utility gained by u because of getting matched to v and the right side quantifies the utility lost by u because of u and v breaking their present matchings with w and z respectively. Hence, Inequality (1) implies that u gains more utility by getting matched with v than it loses because of u and v breaking their matchings with v and z. Inequality (2) can similarly be explained in the context of node v.

Definition 2. *We call an improving deviation a* **swivel** *whenever two neighbors get matched such that at least one node among the two neighbors was not matched before the deviation.*

See Figure 2 for explanation. For such a swivel to occur, the following set of conditions must hold.

$$(1+\alpha_1)r_{uv} > (1+\alpha_1)r_{uw} \tag{3}$$

$$(1+\alpha_1)r_{uv} > (\alpha_1+\alpha_{vw})r_{uw} \tag{4}$$

Inequality (3) says that u gains more utility by getting matched with v than it loses by breaking its matching with w. Inequality (4) says that v gains more utility by getting matched with u than the utility it loses because of u breaking its matching with w. As $\alpha_1 + \alpha_{vw} \leq 1 + \alpha_1$, Inequality (4) is implied by Inequality (3). This means that if v is unmatched, the only condition for (u, v) to be a blocking pair is that u should have net increase in utility by getting matched with v. This is true even if v and w are neighbors. Canceling the factor of $1 + \alpha_1$, we can thus summarize this (necessary and sufficient) condition for swivel to be an improving deviation as:

$$r_{uv} > r_{uw} \tag{5}$$

All improving deviations by a blocking pair can be classified as either a biswivel or a swivel, depending only on whether both nodes are matched or not. The following observation will later be useful.

Lemma 1. *Suppose nodes u and w are matched in M. If (u, v) forms a blocking pair, then $r_{uv} > r_{uw}$.*

Proof. Straightforward with inequalities (1) and (2) for a biswivel and inequality (5) for a swivel. □

2.1 Existence and Welfare of Stable Matchings with Friendship Utilities

Theorem 1. *A stable matching exists in stable matching games with friendship utilities. Moreover, the set of stable matchings without friendship (i.e., when $\vec{\alpha} = \mathbf{0}$) is a subset of the set of stable matchings with friendship utilities on the same graph.*

Proof. If $\vec{\alpha} = \mathbf{0}$, a stable matching M exists, because in this special case our model is correlated stable matching. We prove now is that the same M is stable when we have friendship utilities.

Suppose for contradiction that M is unstable for some value of $\vec{\alpha}$. This is possible only if we have a blocking pair (u, v). But this cannot happen because:

- If both u and v were unmatched in M then M could not have been stable for $\vec{\alpha} = \mathbf{0}$.
- If exactly one of u and v is unmatched in M, say u is matched to w and v is unmatched, then for (u, v) to be a blocking pair, $r_{uv} > r_{uw}$ by Lemma 1. But in such a case, M could not have been stable for $\vec{\alpha} = \mathbf{0}$.

- Suppose both u and v are matched in M, say u is matched to w and v is matched to z. In such a case if (u, v) forms a blocking pair corresponding to a biswivel, then by Lemma 1, we have $r_{uv} > r_{uw}$ and $r_{uv} > r_{vz}$ and thus M could not have been stable for $\vec{\alpha} = \mathbf{0}$.

Hence we have shown that no blocking pair exists in M with friendship utilities, thus proving the theorem. □

Theorem 2. *The price of anarchy in stable matching games with friendship utilities is at most 2, and this bound is tight.*

Proof. This theorem is simply a special case of our much more general Theorem 6, which proves a price of anarchy bound of $1 + \frac{R+\alpha_1}{1+\alpha_1 R}$, with R being a measure of how unequally players can share rewards on an edge. When players share edge rewards equally, the price of anarchy bound in Theorem 6 reduces to $1 + \frac{1+\alpha_1}{1+\alpha_1} = 2$, as desired. To show that this bound is tight, simply consider a 3-edge path with all edge rewards being 1, for any value of $\vec{\alpha}$. □

2.2 Price of Stability and Convergence

The main result in this section bounds the price of stability in stable matching games with friendship utilities to $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$, and this bound is tight (see Theorem 4 below). This bound has some interesting implications. It is decreasing in each α_1 and α_2, hence having friendship utilities always yields a lower price of stability than without friendship utilities. Also, note that values of $\alpha_3, \alpha_4, ..., \alpha_{diam(G)}$ have no influence. Thus, caring about players more than distance 2 away does not improve the price of stability in any way. Also, if $\alpha_1 = \alpha_2 = 1$, then PoS = 1, i.e., there will exist a stable matching which will also be a social optimum. Thus *loving thy neighbor and thy neighbor's neighbor but nobody beyond* is sufficient to guarantee that there exists at least one socially optimal stable matching. In fact, due to the shape of the curve, even small values of friendship quickly decrease the price of stability; e.g., setting $\alpha_1 = \alpha_2 = 0.1$ already decreases the price of stability from 2 to ~ 1.7.

We will establish the price of stability bound by defining an algorithm that creates a good stable matching in polynomial time. One possible idea to create a stable matching that is close to optimum is to use a Best-Blocking-Pair algorithm: start with the best possible matching, i.e. a social optimum, which may or may not be stable. Now choose the "best" blocking pair (u, v): the one with maximum edge reward r_{uv}. Allow this blocking pair to get matched to each other instead of their current partners. Check if the resulting matching is stable. If it is not stable then allow the best blocking pair for this matching to get matched. Repeat the procedure until there are no more blocking pairs, thereby obtaining a stable matching.

This algorithm gives the desired price of stability and running time bounds for the case of "altruism" when all α_i are the same, see Corollary 1 below. To provide the desired bound with general friendship utilities, we must alter this algorithm slightly using the concept of *relaxed* blocking pair.

Definition 3. *Given a matching M, we call a pair of nodes (u, v) a relaxed blocking pair if either (u, v) form an improving swivel, or u and v are matched to w and z respectively, with the following inequalities being true:*

$$(1 + \alpha_1)r_{uv} > (1 + \alpha_1)r_{uw} + (\alpha_1 + \alpha_2)\, r_{vz} \tag{6}$$
$$(1 + \alpha_1)r_{uv} > (1 + \alpha_1)r_{vz} + (\alpha_1 + \alpha_2)\, r_{uw} \tag{7}$$

In other words, a relaxed blocking pair ignores the possible edges between nodes u and z, and has α_2 in the place of α_{uz} (similarly, α_2 in the place of α_{vw}). It is clear from this definition that a

1. `Initialize` $M = M^*$ `where` M^* `is a socially optimum matching.`

2. `If there is no relaxed blocking pair, terminate. Otherwise, resolve relaxed blocking pair` (u, v) `with maximum edge reward` r_{uv} `by adding` (u, v) `to` M `and removing from` M `any other matching edges incident to` u `and` v`.`

3. `Repeat step 2.`

Figure 3: BEST-RELAXED-BLOCKING-PAIR Algorithm

blocking pair is also a relaxed blocking pair, since the conditions above are less constraining than Inequalities (1) and (2). Thus a matching with no relaxed blocking pairs is also a stable matching. Moreover, it is easy to see that Lemma 1 still holds for relaxed blocking pairs. We will call a relaxed blocking pair satisfying Inequalities (6) and (7) a *relaxed biswivel,* which may or may not correspond to an improving deviation, since a relaxed blocking pair is not necessarily a blocking pair.

2.2.1 The BEST-RELAXED-BLOCKING-PAIR Algorithm

Our algorithm to compute a near-optimal stable matching is the BEST-RELAXED-BLOCKING-PAIR algorithm shown in Fig.3. To establish the efficient running time and the price of stability bound of the resulting stable matching, we first analyze the dynamics of this algorithm and prove some helpful lemmas. We can interpret the algorithm as a sequence of swivel and relaxed biswivel deviations, each inserting one edge into M, and removing up to two edges. It is not guaranteed that the inserted edge will stay forever in M, as a subsequent deviation can remove this edge from M. Let $O_1, O_2, O_3, \cdots$ denote this sequence of deviations, and $e(i)$ denote the edge which got inserted into M because of O_i. We analyze the dynamics of the algorithm in the following two lemmas.

Lemma 2. *The first deviation O_1 during the execution of* BEST-RELAXED-BLOCKING-PAIR *is a relaxed biswivel.*

Proof. Having O_1 as a swivel will strictly improve the value of matching by Lemma 1. Hence if we begin the algorithm with $M = M^*$, having O_1 as a swivel will produce a matching with value strictly greater than M^*, which is a contradiction. □

Lemma 3. *Let O_j be a relaxed biswivel that takes place during the execution of the best relaxed blocking pair algorithm. Suppose a deviation O_k takes place before O_j. Then we have $r_{e(k)} \geq r_{e(j)}$. Furthermore, if O_k is a relaxed biswivel then $e(k) \neq e(j)$ (thus at most $|E(G)|$ relaxed biswivels can take place during the execution of the algorithm).*

It is important to note that this lemma does *not* say that $r_{e(i)} \geq r_{e(j)}$ for $i < j$. We are only guaranteed that $r_{e(i)} \geq r_{e(j)}$ for $i < j$ if O_j is a *relaxed biswivel.* Between two successive relaxed biswivels O_k and O_j, the sequence of $r_{e(i)}$ for consecutive swivels can and does increase as well as decrease, and the same edge may be added to the matching multiple times. All that is guaranteed is that $r_{e(j)}$ for a biswivel O_j will have a lower value than all the preceding $r_{e(i)}$'s. Thus, this lemma suggests a nice representation of BEST-RELAXED-BLOCKING-PAIR in terms of phases, where we define a *phase* as a subsequence of deviations that begins with a relaxed biswivel and continues until the next relaxed biswivel. Lemma 2 shows that the start of the sequence is also the start of the first phase. Lemma 3 guarantees that at the start of each phase, the $r_{e(j)}$ value is smaller than the values in all previous phases, and that there is only a polynomial number of phases. Now we proceed to prove Lemma 3.

Proof. Let $e(j) = (v, z)$ get inserted in M because of a relaxed biswivel O_j. We first give a brief outline of the proof. Suppose that the claim $r_{e(k)} \geq r_{e(j)}$ for $k < j$ is false and we have an O_k with $k < j$ such that $r_{e(k)} < r_{e(j)}$. Clearly (v, z) could not have been a relaxed blocking pair just before O_k, otherwise the algorithm would have chosen (v, z) as the best relaxed blocking pair instead of O_k. We will show that this leads to a conclusion that (v, z) cannot be a relaxed blocking pair even for O_j. This is a contradiction, hence our assumption of $r_{e(k)} < r_{e(j)}$ could not have been correct. Thus for all O_k such that $k < j$ we will have $r_{e(k)} \geq r_{e(j)}$. Later we will use similar reasoning to prove that if O_i with $i < j$ is a relaxed biswivel that takes place before a relaxed biswivel O_j then $e(i) \neq e(j)$. Now let us proceed to the proof.

Suppose to the contrary that we have O_k with $k < j$ such that $r_{e(k)} < r_{e(j)}$ with O_j being a relaxed biswivel. As discussed in the outline of the proof, this implies that (v, z) was not a relaxed blocking pair at the time O_k was selected. Let S be the set of nodes with whom v and z are matched at the time that O_k is selected. As long as S does not change, v and z will not be a relaxed blocking pair, since the change in utility experienced by v and z from matching to each other depends only on their partners in the current matching, i.e., the set S. Thus for the relaxed biswivel O_j to occur, S must change between O_k and O_j. We will show that this leads to a contradiction: that (v, z) cannot be a relaxed blocking pair for the time O_j is selected.

Suppose v is matched to x and z is matched to y just before biswivel O_j. Since (v, z) is a relaxed blocking pair at this point, we thus have

$$(1 + \alpha_1) r_{vz} > (1 + \alpha_1) r_{vx} + (\alpha_1 + \alpha_2) r_{zy} \tag{8}$$

$$(1 + \alpha_1) r_{vz} > (1 + \alpha_1) r_{zy} + (\alpha_1 + \alpha_2) r_{vx}. \tag{9}$$

Recall that (v, z) was not a relaxed blocking pair just before O_k, and to make it a relaxed blocking pair for O_j, S must change between O_k and O_j. Let O_l be the last deviation which changed S to $\{x, z\}$. Without loss of generality, we can assume that O_l adds the edge (v, x). Now we have two cases:

- (v, z) was a relaxed blocking pair at the time O_l is selected: in this case (v, x) could not have been the best relaxed blocking pair for O_l because inequality (8) tells us $r_{vz} > r_{vx}$.

- (v, z) was not a relaxed blocking pair at the time O_l is selected: Suppose v was matched with w before O_l. As (v, z) was not a relaxed blocking pair just before O_l we have

$$\text{Either } (1 + \alpha_1) r_{vz} \leq (1 + \alpha_1) r_{vw} + (\alpha_1 + \alpha_2) r_{zy} \tag{10}$$

$$\text{OR } (1 + \alpha_1) r_{vz} \leq (1 + \alpha_1) r_{zy} + (\alpha_1 + \alpha_2) r_{vw} \ . \tag{11}$$

  (If v was unmatched just before O_l, then substitute $r_{vw} = 0$ to obtain the appropriate condition.) Assume that it is inequality (10) that holds. Then, because O_l removes edge (v, w) and adds edge (v, x), we have $r_{vx} > r_{vw}$ as Lemma 1 holds for relaxed blocking pairs. Thus, it holds

$$(1 + \alpha_1) r_{vz} \leq (1 + \alpha_1) r_{vx} + (\alpha_1 + \alpha_2) r_{zy} \ . \tag{12}$$

  This contradicts inequality (8), and thus (v, z) cannot be a relaxed blocking pair at the time O_j is selected. The same conclusion can be reached if we assume inequality (11) holds true.

In either case we obtain a contradiction, thus showing that if O_j is a relaxed biswivel, then for all O_k with $k < j$, we have $r_{e(k)} < r_{e(j)}$.

The only remaining piece is to prove $e(k) \neq e(j)$ if O_k is a relaxed biswivel. Notice that if $e(k) = e(j) = (v, z)$, then S has to change between O_k and O_j. Now we use exactly the reasoning from the previous paragraph to arrive at a contradiction, thus proving that $e(k) \neq e(j)$. □

If $\alpha_1 = \alpha_2$, the conditions for a blocking pair are identical to the conditions for a relaxed blocking pair. Hence, our algorithm corresponds to letting the best blocking pair deviate at each step. As a special case, for $\vec{\alpha} = \mathbf{0}$ and correlated stable matching, this algorithm is known to provide a stable matching in polynomial time [4]. For friendship utilities, however, (quick) convergence was previously unknown. We now show that even with the addition of friendship, Best-Relaxed-Blocking-Pair (and thus Best-Blocking-Pair when $\alpha_1 = \alpha_2$) terminates and produces a stable matching. Moreover, it does so in polynomial time.

If instead of the best we pick some arbitrary blocking pair, then there exists an instance in which, starting from the empty matching, a sequence of blocking pairs of length $2^{\Omega(n)}$ exists until reaching a stable matching, even without friendship. This is directly implied by recent results in correlated stable matching [22].

A trivial adjustment of the gadget in [22] allows us to construct the exponential sequence even when starting from the social optimum. We scale the reward of each (original) edge $i \in \{1, \ldots, m\}$ in the gadget from i to $1 + i \cdot \epsilon$, for some tiny $\epsilon > 0$. This preserves all incentives and the structure of all blocking pairs. Then, we add an auxiliary neighbor for each (original) player and connect it via an auxiliary edge of reward 1. The social optimum is obviously given by matching each original player with his auxiliary neighbor. However, the exponential sequence of blocking pairs still exists, because auxiliary edges are not rewarding enough to influence blocking pairs among original players. Given that such exponential-length sequences exist, it is perhaps surprising that our algorithm indeed finds a stable matching and it terminates in polynomial time.

Theorem 3. Best-Relaxed-Blocking-Pair *outputs a stable matching after $O(m^2)$ iterations, where m is the number of edges in the graph.*

Proof. Consider the three possible changes that can occur to the matching M during each iteration: a swivel could add a new edge, or it could delete an edge and add an edge with strictly higher r_e value. A relaxed biswivel deletes two edges and adds an edge with higher r_e value than either. If no biswivels take place, the number of edges in the matching cannot decrease. Hence, the total number of consecutive swivels is $O(m^2)$. Now consider the structure of the sequence in phases. The relaxed biswivel in the beginning can allow at most m additional swivels to occur, since it reduces the number of edges by one. As there are at most m relaxed biswivel deviations possible by Lemma 3, the algorithm terminates after at most $O(m^2)$ deviations. At the end of the algorithm, there exist no relaxed blocking pairs. Since a blocking pair is also a relaxed blocking pair, the final matching produced by the algorithm is a stable matching. □

As we can have only a polynomial number of consecutive swivel deviations between each relaxed biswivel, we know that every phase (defined as a maximal subsequence of consecutive swivels) lasts only a polynomial amount of time, and there are only $O(m)$ phases by Lemma 3. Moreover, in each phase, the value of the matching only increases, since swivels only remove an edge if they add a better one. Below, we use the fact that only relaxed biswivel operations reduce the cost of the matching to bound the cost of the stable matching this algorithm produces.

2.2.2 Upper Bound on Price of Stability

To prove the bound, we need some notation and useful lemmas. We define a sequence of mappings from M^* to $E(G)$. Define $h_0 : M^* \to E(G)$ as $h_0(e) = e$. Depending on O_i, we define h_i as follows: Suppose O_i is a deviation that removes edge $h_{i-1}(e_j)$ from M. If O_i inserts edge e_l in M then set $h_i(e_j) = e_l$. For all other $e_k \in M^*$, keep $h_i(e_k)$ same as $h_{i-1}(e_k)$. Let us note that a deviation O_i may not remove any edges from $\{h_{i-1}(e_j) : e_j \in M^*\}$. This can happen because during the course of the algorithm, two unmatched nodes can get matched, say to insert e_p into M. No edges in M^* get mapped to e_p. If this edge is removed from M by a later deviation, the mapping may not change, since no edge is mapped to e_p. To summarize, h_i may be the same as h_{i-1}, or may differ from h_{i-1} in one location (in case of a swivel), or in two locations (in case of a relaxed biswivel). Denote the resulting mapping when our algorithm terminates by h_M.

Coupling Lemma 1 with the definition of mappings h_i, we directly see:

Lemma 4. *$\{r_{h_i(e)}\}_{i\geq 0}$ is a nondecreasing sequence and $r_{h_{i+1}(e)} > r_{h_i(e)}$ whenever $h_{i+1}(e) \neq h_i(e)$.*

The next lemma will be instrumental in proving the price of stability bound.

Lemma 5. *If $h_M(e_i) = h_M(e_j)$ with $e_i \neq e_j$ then*

1. *There must exist a relaxed biswivel O_k such that $h_{k-1}(e_i) \neq h_{k-1}(e_j)$ but O_k makes $h_k(e_i) = h_k(e_j)$. Furthermore, for all $p \geq k$ we have $h_p(e_i) = h_p(e_j)$.*
2. *There does not exist another $e_l \in M^*$ such that $h_M(e_l) = h_M(e_i) = h_M(e_j)$.*
3. *$r_{e_i} + r_{e_j} < \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \times r_{h_M(e_i)}$*

Proof. To prove the first part, say O_l was the first deviation such that $h_{l-1}(e_i) \neq h_{l-1}(e_j)$ and $h_l(e_i) = h_l(e_j)$. It cannot happen because of a swivel deviation because a swivel can make $h_l(e) \neq h_{l-1}(e)$ for at most for one $e \in M^*$. Thus O_l must be a relaxed biswivel. Set $k = l$ and it is easy to see that for $p \geq k$ we have $h_p(e_i) = h_p(e_j)$. Hence the first part is proven.

To prove the second part, suppose there exists an e_l with $e_l \neq e_i \neq e_j$ such that $h_M(e_l) = h_M(e_i) = h_M(e_j)$. From the first part, there must exist a relaxed biswivel O_k s.t. $h_{k-1}(e_i) \neq h_{k-1}(e_l)$ but $h_k(e_i) = h_k(e_l)$. Similarly there must exist a relaxed biswivel O_p s.t. $h_{p-1}(e_i) \neq h_{p-1}(e_j)$ but $h_p(e_i) = h_p(e_j)$. Without loss of generality say $p > k$. Using Lemma 3 we get $r_{e(k)} \geq r_{e(p)}$. But from Lemma 4, we have $r_{e(k)} < r_{e(p)}$, since $e(p) = h_p(e_i) > h_k(e_i) = e(k)$. We obtain a contradiction here, thus proving that there does not exist another $e_l \in M^*$, with $h_M(e_l) = h_M(e_i) = h_M(e_j)$.

To prove the third part, consider a relaxed biswivel O_k such that $h_{k-1}(e_i) \neq h_{k-1}(e_j)$ and $h_k(e_i) = h_k(e_j)$. Substitute $r_{uv} = r_{h_k(e_i)}$, $r_{uw} = r_{h_{k-1}(e_i)}$ and $r_{vz} = r_{h_{k-1}(e_j)}$ in inequalities (1) and (2). Adding these inequalities and simplifying, we get

$$r_{h_{k-1}(e_i)} + r_{h_{k-1}(e_j)} < \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot r_{h_k(e_i)} \quad . \tag{13}$$

From Lemma 4, we know $\{r_{h_i(e)}\}_{i\geq 0}$ is a nondecreasing sequence. Using this in (13) we get

$$r_{e_i} + r_{e_j} < \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot r_{h_M(e_i)} \quad . \tag{14}$$

□

Using Lemma 5, we can partition edges of M^* into two sets as follows: Let B denote the set of edges $e_i \in M^*$ such that $h_M(e_i) = h_M(e_j)$ for some $e_j \in M^*$ and let A denote the remaining edges in M^*. We can further partition set B into two sets P and Q as follows: choose a pair e_i and e_j in B such that $h_M(e_i) = h_M(e_j)$. Denote e_j by $\mu(e_i)$. Put e_i in P and $\mu(e_i)$ in Q. Notice that value of the matching M that Best-Relaxed-Blocking-Pair gives as output is at least $\sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} r_{h_M(e)}$. Possible additional edges in M are produced because of swivels which match two previously unmatched nodes with each other.

This allows us to prove the main theorem of this section:

Theorem 4. *The price of stability in stable matching games with friendship utilities is at most* $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$*, and this bound is tight.*

Proof. The value of M^* is given by

$$\begin{aligned} w(M^*) &= \sum_{e \in A} r_e + \sum_{e \in P} r_e + \sum_{e \in Q} r_e \\ &= \sum_{e \in A} r_e + \sum_{e \in P} (r_e + r_{\mu(e)}) \ . \end{aligned}$$

Using Lemma 5, for $e \in P$ we have $r_e + r_{\mu(e)} < \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot r_{h_M(e)}$. Using Lemma 4, for $e \in A$ we have $r_e \leq r_{h_M(e)}$. Thus, we get

$$\begin{aligned} w(M^*) &\leq \sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot r_{h_M(e)} \\ &\leq \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot \left(\sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} r_{h_M(e)} \right) \ . \end{aligned}$$

Note that this inequality may not be strict since A may be empty. This could happen if each edge in M^* gets removed because of a relaxed biswivel as the algorithm proceeds (though it may be possible that it is inserted later). We also have $w(M) \geq \sum_{e \in A} r_{h_M(e)} + \sum_{e \in P} r_{h_M(e)}$ for the final matching M of the algorithm. Using this,

$$w(M^*) \leq \frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2} \cdot w(M) \ ,$$

which proves the bound on the price of stability, since M is a stable matching.

To prove the tightness of the bound, assume $\alpha_2 = 0$ and set $r_{uv} = \frac{1+2\alpha_1+\epsilon}{1+\alpha_1}$, $r_{uw} = r_{vz} = 1$ in Fig 1. Then we have $\{(u.v)\}$ as the only stable matching but the social optimum is $\{(u,w),(v,z)\}$. Thus, we get a price of stability of $\frac{2+2\alpha_1}{1+2\alpha_1+\epsilon}$. With $\epsilon \to 0$, this yields a tight bound for $\alpha_2 = 0$. □

Theorems 3 and 4, yield the following corollary about the behavior of best blocking pair dynamics. It applies in particular to the model of altruism when $\alpha_i = \alpha$ for all $i = 1, \ldots, diam(G)$.

Corollary 1. *If $\alpha_1 = \alpha_2$ and we start from the centrally optimum matching,* Best-Blocking-Pair *converges in $O(m^2)$ time to a stable matching that is at most a factor of $\frac{2+2\alpha_1}{1+2\alpha_1+\alpha_2}$ worse than the optimum.*

Proof. When $\alpha_1 = \alpha_2$, Best-Relaxed-Blocking-Pair is Best-Blocking-Pair. □

3 Stable Matching with Friendship and General Reward Sharing

In the previous section we assumed that for $(u, v) \in M$ both u and v get the *same reward* r_{uv}. Let us now treat the mode general case where u and v receive different rewards for $(u, v) \in M$. We define r^x_{xy} as the reward of x from edge $(x, y) \in M$. We interpret our model in a reward sharing context, where x and y share a total reward of $r_{xy} = r^x_{xy} + r^y_{xy}$. The correlated matching model of Section 2 can equivalently be formulated as equal sharing with nodes u and v receiving a reward of $r_{uv}/2$.

Let us again write explicit conditions for nodes to form a blocking pair in this context and define some helpful notation. The necessary and sufficient conditions for nodes (u, v) to form a biswivel from nodes w and z (See Fig. 1) in reward sharing with friendship are:

$$\begin{aligned} r^u_{uv} + \alpha_1 r^v_{uv} &> r^u_{uw} + \alpha_1(r^w_{uw} + r^v_{vz}) + \alpha_{uz} r^z_{vz} \\ r^v_{uv} + \alpha_1 r^u_{uv} &> r^v_{vz} + \alpha_1(r^z_{vz} + r^u_{uw}) + \alpha_{vw} r^w_{uw}. \end{aligned}$$

We define $q^x_{xy} = r^x_{xy} + \alpha_1 r^y_{xy}$. Then the conditions for biswivel such as shown in Fig. 1 are:

$$q^u_{uv} > q^u_{uw} + \alpha_1 r^v_{vz} + \alpha_{uz} r^z_{vz} \tag{15}$$

$$q^v_{uv} > q^v_{vz} + \alpha_1 r^u_{uw} + \alpha_{vw} r^w_{uw}. \tag{16}$$

Similarly, the necessary and sufficient conditions for swivel (See Fig. 2) are

$$\begin{aligned} r^u_{uv} + \alpha_1 r^v_{uv} &> r^u_{uw} + \alpha_1 r^w_{uw} \\ r^v_{uv} + \alpha_1 r^u_{uv} &> \alpha_1 r^u_{uw} + \alpha_{vw} r^w_{uw}. \end{aligned}$$

Using the definition of $q^x_{xy}(\cdot, \cdot)$, the conditions for swivel become:

$$q^u_{uv} > q^u_{uw} \tag{17}$$

$$q^v_{uv} > \alpha_1 r^u_{uw} + \alpha_{vw} r^w_{uw} \tag{18}$$

Let us define $q_{xy} = q^x_{xy} + q^y_{xy}$. Thus we obtain $q_{xy} = (1 + \alpha_1) r_{xy}$.

3.1 Existence of a Stable Matching

Without friendship utilities, our stable matching game reduces to the stable roommates problem (i.e., non-bipartite stable matching), since reward shares can be arbitrary and thus induce arbitrary preference lists for each node. It is well known that a stable matching may not exist in instances of the stable roommates problem [19]. While we are able to prove existence of integral stable matching for several interesting special cases (see Section 3.4 below), the addition of friendship further complicates matters. In Section 2.1 we showed that for equal sharing, a stable matching without friendship utilities (i.e., $\vec{\alpha} = \mathbf{0}$) is also a stable matching when we have friendship utilities. This is no longer true for unequal reward sharing: adding a social context can completely change the set of stable matchings. In Section 3.5 we give such examples, including an example where adding a social context (i.e., increasing $\vec{\alpha}$ above zero) destroys all stable matchings that exist when $\vec{\alpha} = \mathbf{0}$.

Although stable matchings may not exist in general non-bipartite graphs, *fractional* stable matchings are guaranteed to exist [2]. Fortunately, as we prove below, this holds even in the presence of friendship utilities with general reward sharing: A fractional stable matching always exists.

A fractional stable matching is a fractional matching without blocking pairs. Specifically, a biswivel occurs when there is an edge (u, v) such that increasing the strength of edge (u, v), and decreasing the strength of some other edges (u, w) and (v, z) would strictly improve the utilities of both u and v. The inequalities that would make this be true are exactly (15) and (16); they do not change simply due to the fractional nature of the matching (note, however, that for a biswivel to make sense, it is necessary that $x_{uv} < 1$, $x_{uw}, x_{vz} > 0$). Similarly, a swivel occurs when increasing the strength of an edge (u, v) with node v not being tight (i.e., $\sum_{e \ni v} x_e < 1$), and decreasing the strength of some edge (u, w) would strictly improve the utilities of both u and v; or when there are two adjacent nodes that are not tight. The inequalities that would make this be true are exactly (17) and (18).

Theorem 5. *A fractional stable matching always exists, even in the case of friendship utilities and general reward sharing.*

Proof. We denote by SRP_q the instance of the stable roommates problem where we have exactly the same edges in the graph as in our network. However, in SRP_q the nodes prepare their preference lists based on q^x_{xy}, i.e., a node u will prefer node v as roommate over w iff $q^u_{uv} > q^u_{uw}$, breaking ties arbitrarily. Note that SRP_p can be seen as an instance of the ordinary stable roommates problem without friendships.

SRP_q has at least one fractional (and, in fact, half-integral) stable matching [2]. We will now show that a fractional stable matching for SRP_q is a fractional stable matching for our matching game with unequal reward sharing and friendship utilities as well.

Suppose a fractional stable matching M for SRP_q is not a fractional stable matching for unequal reward sharing with friendship utilities. Then there exists a blocking pair (u, v) with one of the following two possibilities:

- (u, v) forms a biswivel, so the inequalities (15) and (16) must hold true. These inequalities imply $q^u_{uv} > q^u_{uw}$ and $q^v_{uv} > q^v_{vz}$. But then (u, v) would be a blocking pair in SRP_q. This contradicts that M is stable in SRP_q.
- (u, v) forms a swivel, say with v such that $\sum_{e \ni v} x_e < 1$ and with u such that $\sum_{e \ni u} x_e = 1$. (It cannot be that both u and v are not tight, since otherwise M would not be stable in SRP_q.) Then for (u, v) to be a blocking pair inequalities (17) and (18) must hold true. But these inequalities imply $q^u_{uv} > q^u_{uw}$ and thus (u, v) would be a blocking pair in SRP_q. This contradicts that M is stable in SRP_q.

Hence, M must be stable with unequal reward sharing and friendship utilities. Moreover, the set of fractional stable matchings in SRP_q is a subset of the set of fractional stable matchings in unequal reward sharing with friendship utilities. Since there exists at least one fractional stable matching in SRP_q, the theorem is proved. □

3.2 Price of Anarchy with General Reward Sharing

In this section we prove tight bounds for the price of anarchy of stable matching with friendship utilities in the presence of general reward sharing. Since an integral stable matching may not exist, we instead consider fractional matching; by price of anarchy here we mean the ratio of the total reward in a socially optimum *fractional* matching with the worst *fractional* stable matching. The corresponding ratio between the integral versions is trivially upper bounded by this amount as well.

We define R as

$$R = \max_{(u,v) \in E(G)} \frac{r^u_{uv}}{r^v_{uv}} \tag{19}$$

Note that we will always have $R \geq 1$. By definition of q, we also have

$$\frac{q_{xy}^x}{q_{xy}^y} = \frac{r_{xy}^x + \alpha_1 r_{xy}^y}{r_{xy}^y + \alpha_1 r_{xy}^x}$$

Using the fact that $\frac{p+\alpha_1}{1+\alpha_1 p}$ is an increasing function of p and using the definition of R, we thus obtain

$$\frac{q_{xy}^x}{q_{xy}^y} \leq \frac{R+\alpha_1}{1+\alpha_1 R} \tag{20}$$

Most of this section is devoted to proving the following theorem:

Theorem 6. *The (fractional) price of anarchy for general reward sharing with friendship utilities is at most* $1+Q$*, where* $Q = \max_{(u,v)\in E(G)} \frac{q_{uv}^u}{q_{uv}^v} \leq \frac{R+\alpha_1}{1+\alpha_1 R}$*, and this bound is tight.*

Let us quickly consider the implications of this bound. If $R = 1$, the bound is 2. This result implies Theorem 2, since when we have $R = 1$, then both u and v get the same reward from an edge $(u, v) \in M$. If $\alpha_1 = 0$, the bound is $1 + R$. The tightness of this bound implies that as sharing becomes more unfair, i.e., as $R \to \infty$, we can find instances where the price of anarchy is unbounded. Unequal sharing can make things much worse for the stable matching game.

Notice, however, that $\frac{R+\alpha_1}{1+\alpha_1 R}$ is a decreasing function of α_1. As α_1 goes from 0 to 1, the bound goes from $1 + R$ to 2. Without friendship utilities ($\vec{\alpha} = \mathbf{0}$), we have a tight upper bound of $1 + R$, which is extremely bad for large R. As α_1 tends to 1, however, the price of anarchy drops to 2, independent of R. For example, for $\alpha_1 = 1/2$ it is only 3. Thus, social context can drastically improve the outcome for the society, especially in the case of unfair and unequal reward sharing.

Now let us proceed to the proof of Theorem 6.

Proof. We first introduce some notation. We denote by M^* an optimum fractional matching and use M to denote a fractional or integral stable matching. Let x_{uv}^* (or x_{uv}) denote the fraction of edge (u, v) present in M^* (or M). Furthermore,

$$\begin{aligned}
S &= \{e \in E(G) : x_e^* > x_e\} \\
T &= \{e \in E(G) : x_e^* \leq x_e\} \\
S_u &= \{e \in E(G) : e \text{ is incident on } u \text{ and } e \in S\} \\
T_u &= \{e \in E(G) : e \text{ is incident on } u \text{ and } e \in T\} \\
E_u &= \text{All edges incident on } u\text{. Note that } E_u = S_u \cup T_u \\
y_u &= 1 - \sum_{(u,v)\in E_u} x_{uv} \\
y_u^* &= 1 - \sum_{(u,v)\in E_u} x_{uv}^* \\
\Delta_u &= \sum_{(u,v)\in T_u} (x_{uv} - x_{uv}^*) + \max\{(y_u - y_u^*), 0\}
\end{aligned}$$

The idea of the proof is the following. Consider an edge $(u, v) \in S$. If the fraction of the edge (u, v) in M was increased to x_{uv}^* from x_{uv} by decreasing fractions of some other edges in M incident on u and v, then at least one of the endpoints of (u, v) does not improve its utility. Tag this endpoint as corresponding to edge (u, v) and denote the set of tagged nodes by B. We get one inequality for each node $u \in B$ for such a modification of fractions of adjoining edges. We will show that adding all such inequalities gives us the following:

$$\sum_{(u,v)\in S, u\in B} q^u_{uv} \cdot (x^*_{uv} - x_{uv}) \;\; \leq \;\; \sum_{u\in B} \sum_{(u,w)\in T_u} q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \;\; . \tag{21}$$

We prove Eqn (21) separately below. Now, as $q^u_{uv} \geq q_{uv}/(1+Q)$ and $q_{uw} = q^u_{uw} + q^w_{uw}$, Eqn (21) becomes

$$\sum_{(u,v)\in S} \frac{1}{1+Q} q_{uv} \cdot (x^*_{uv} - x_{uv}) \;\; \leq \;\; \sum_{(u,w)\in T} q_{uw} \cdot (x_{uw} - x^*_{uw}) \;\; .$$

Using algebraic simplifications and the fact that we have $1 \geq 1/(1+Q)$, we get the following sequence of inequalities:

$$\begin{aligned}
\sum_{(u,v)\in S} \frac{1}{1+Q} \cdot q_{uv} \cdot x^*_{uv} + \sum_{(u,v)\in T} q_{uv} \cdot x^*_{uv} \;\; &\leq \;\; \sum_{(u,w)\in T} q_{uw} \cdot x_{uw} + \sum_{(u,w)\in S} \frac{1}{1+Q} \cdot q_{uw} \cdot x_{uw} \\
\Rightarrow \quad \sum_{(u,v)\in S\cup T} \frac{1}{1+Q} \cdot q_{uv} \cdot x^*_{uv} \;\; &\leq \;\; \sum_{(u,w)\in S\cup T} q_{uw} \cdot x_{uw} \\
\Rightarrow \quad \frac{\sum_{(u,v)\in G} q_{uv} \cdot x^*_{uv}}{\sum_{(u,w)\in G} q_{uw} \cdot x_{uw}} \;\; &\leq \;\; 1+Q \;\; ,
\end{aligned} \tag{22}$$

where for the last inequality we have used the fact that $S \cup T$ covers all the edges in the graph. This proves the claim.

It remains to prove Eqn (21). Suppose node u gets tagged for edge (u,v) if we increase x_{uv} to x^*_{uv} by doing the following two steps:

- Decrease fraction of each $(u,w) \in T_u$ by $(x^*_{uv} - x_{uv})(x_{uw} - x^*_{uw})/\Delta_u$ and decrease fraction of each $(v,z) \in T_v$ by $(x^*_{uv} - x_{uv})(x_{vz} - x^*_{vz})/\Delta_v$ AND
- If $y_u > y^*_u$ then decrease y_u by $(x^*_{uv} - x_{uv})(y_u - y^*_u)/\Delta_u$ and if $y_v > y^*_v$ then decrease y_v by $(x^*_{uv} - x_{uv})(y_v - y^*_v)/\Delta_v$.

If the utility of u does not improve then

$$\begin{aligned}
q^u_{uv} \cdot (x^*_{uv} - x_{uv}) \;\; \leq \;\; & \sum_{(u,w)\in T_u} q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \cdot c^{uv}_u \\
& + \sum_{(u,w)\in T_v} \alpha_1 r^v_{vz} \cdot (x_{vz} - x^*_{vz}) \cdot c^{uv}_v \\
& + \sum_{(u,w)\in T_v} \alpha_{uz} r^z_{vz} \cdot (x_{vz} - x^*_{vz}) \cdot c^{uv}_v \;\; ,
\end{aligned} \tag{23}$$

where $c^{uv}_u = (x^*_{uv} - x_{uv})/\Delta_u$ and likewise for c^{uv}_v. Eqn (23) can be explained as follows. $q^u_{uv} \cdot (x^*_{uv} - x_{uv})$ denotes the utility gained by u on edge (u,v) by increasing x_{uv} to x^*_{uv}. The term $q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \cdot c^{uv}_u$ denotes the utility lost by u because of decreasing x_{uw} for an edge $(u,w) \in T_u$. When x_{vz} decreases for an edge $(v,z) \in T_v$ then by virtue of friendship with v, node u loses $\alpha_1 r^v_{vz} \cdot (x_{vz} - x^*_{vz}) \cdot c^{uv}_v$. When x_{vz} decreases for an edge $(v,z) \in T_v$ then depending on α_{uz} node u loses $\alpha_1 r^z_{vz} \cdot (x_{vz} - x^*_{vz}) \cdot c^{uv}_v$. Note that decreasing y_u is important when x_{uv} cannot be increased to x^*_{uv} without decreasing y_u.

As $\alpha_1 r^v_{vz} + \alpha_{uz} r^z_{vz} \le r^v_{vz} + \alpha_1 r^z_{uz} = q^v_{vz}$, Eqn (23) can be simplified to

$$q^u_{uv} \cdot (x^*_{uv} - x_{uv}) \quad \le \quad \sum_{(u,w)\in T_u} q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \cdot c^{uv}_u + \sum_{(u,w)\in T_v} q^v_{vz} \cdot (x_{vz} - x^*_{vz}) \cdot c^{uv}_v \ . \qquad (24)$$

We can form one such inequality for all edges $(u,v) \in S$. Let us inspect the coefficient of a term $q^u_{uw}(x_{uw} - x^*_{uw})$ appearing on right hand side if we add all these inequalities. Notice that $q^u_{uw}(x_{uw} - x^*_{uw})c^{uv}_u$ appears only once for each edge $(u,v) \in S$ adjoining u. These are precisely the edges in S_u. Thus the coefficient of $q^u_{uw}(x_{uw} - x^*_{uw})$ will be $\sum_{(u,v)\in S_u} c^u_{uv}$ if we add all the inequalities like Eqn (24). By definition of c^u_{uv}, we have $\sum_{(u,v)\in S_u} c^u_{uv}$ is at most 1. Thus a term $q^u_{uw}(x_{uw} - x^*_{uw})$ can appear with coefficient at most 1 if we add all these inequalities like Eqn (24). Thus we get,

$$\sum_{(u,v)\in S, u\in B} q^u_{uv} \cdot (x^*_{uv} - x_{uv}) \quad \le \quad \sum_{u\in G} \sum_{(uw)\in T_u} q^u_{uw} \cdot (x_{uw} - x^*_{uw}) \ .$$

We have proved Eqn (21) which in turn proves our claim.
Tightness of the bound: Consider the 3-length path as shown in Fig. 1. Set $\alpha_2 = \alpha_3 = \cdots = 0$ and use the following values:

$$r^u_{uv} = \frac{1}{1+\alpha_1} \qquad r^v_{uv} = \frac{1}{1+\alpha_1}$$
$$r^u_{uw} = \frac{1}{1+\alpha_1 R} \qquad r^w_{uw} = \frac{R}{1+\alpha_1 R}$$
$$r^v_{vz} = \frac{1}{1+\alpha_1 R} \qquad r^z_{vz} = \frac{R}{1+\alpha_1 R}$$

Note that as desired, $\max_{(x,y)\in E(G)} \frac{r^x_{xy}}{r^y_{xy}} = R$. Using $q^x_{xy} = r^x_{xy} + \alpha r^y_{xy}$, we obtain

$$q^u_{uv} = 1 \qquad q^v_{uv} = 1$$
$$q^u_{uw} = 1 \qquad q^w_{uw} = Q$$
$$q^v_{vz} = 1 \qquad q^z_{vz} = Q$$

Note that as desired, $\max_{(x,y)\in E(G)} \frac{q^x_{xy}}{q^y_{xy}} = \frac{R+\alpha_1}{1+\alpha_1 R} = Q$. We have $\{(u,v)\}$ as a stable matching: Given this matching, (u,w) is not a blocking pair, because $q^u_{uw} \le q^u_{uv}$. Similarly (v,z) too is not a blocking pair in matching $\{(u,v)\}$. Another stable matching is $\{(u,w),(v,z)\}$: Given this matching, (u,v) will not be a blocking pair, becasue $q^u_{uv} < q^u_{uw} + \alpha_1 r^v_{vz}$, and so the condition in inequality (15) is violated. Since there are no other stable matchings for this graph, the price of anarchy will be determined by the value of the worst stable matching which is $\{(u,v)\}$. It is given by

$$\frac{r_{uw} + r_{vz}}{r_{uv}} = \frac{q_{uw} + q_{vz}}{q_{uv}} = 1 + Q \ ,$$

proving tightness of our bound. □

3.3 Price of Stability with General Reward Sharing

In this section, we give a simple lower bound Q' on the price of stability for stable matching games with friendship and reward sharing. Furthermore, we show that this bound is within an additive factor of 1 of optimum, i.e., $Q < Q' \le \text{PoS} \le 1 + Q$.

To prove the lower bound, consider the 3-length path as shown in Fig. 1. Set $\alpha_2 = \alpha_3 = \cdots = 0$ and use the following values:

$$r^u_{uv} = \frac{1}{1+\alpha_1}\left(\frac{1+\alpha_1(R+1)}{(1+\alpha_1 R)} + \epsilon\right) \qquad r^v_{uv} = \frac{1}{1+\alpha_1}\left(\frac{1+\alpha_1(R+1)}{(1+\alpha_1 R)} + \epsilon\right)$$

$$r^u_{uw} = \frac{1}{1+\alpha_1 R} \qquad r^w_{uw} = \frac{R}{1+\alpha_1 R}$$

$$r^v_{vz} = \frac{1}{1+\alpha_1 R} \qquad r^z_{vz} = \frac{R}{1+\alpha_1 R}$$

As desired we have $\max_{(x,y)\in E(G)} \frac{r^x_{xy}}{r^y_{xy}} = R$. Using $q^x_{xy} = r^x_{xy} + \alpha_1 r^y_{xy}$, we obtain

$$q^u_{uv} = \frac{1+\alpha_1(R+1)}{1+\alpha_1 R} \qquad q^v_{uv} = \frac{1+\alpha_1(R+1)}{1+\alpha_1 R}$$

$$q^u_{uw} = 1 \qquad q^w_{uw} = \frac{R+\alpha_1}{1+\alpha_1 R}$$

$$q^v_{vz} = 1 \qquad q^z_{vz} = \frac{R+\alpha_1}{1+\alpha_1 R}$$

As desired, we have $\max_{(x,y)\in E(G)} \frac{q^x_{xy}}{q^y_{xy}} = \frac{R+\alpha_1}{1+\alpha_1 R} = Q$. We have $\{(u,v)\}$ as a stable matching: (u,w) is not a blocking pair, because $q^u_{uw} \le q^u_{uv}$. Similarly (v,z) will not be a blocking pair. Any other fractional matching is no longer stable because (u,v) is a blocking pair as inequalities (15) and (16) are satisfied. However, $\{(u,w),(v,z)\}$ is still the socially optimal matching. Hence, the price of stability is given by

$$\frac{r_{uw}+r_{vz}}{r_{uv}} = \frac{q_{uw}+q_{vz}}{q_{uv}} = \frac{(1+\alpha_1)(1+R)}{1+\alpha_1(R+1)}$$

Let us define $Q' = \frac{(1+\alpha_1)(1+R)}{1+\alpha_1(R+1)}$. The above instance establishes a lower bound on the price of stability of Q', where $Q \le Q' \le Q+1$. $Q+1$ is an upper bound on the price of stability, so our lower bound of Q' is within an additive term of 1 of optimum.

Theorem 7. *The price of stability of stable matching games with friendship and general reward sharing is in* $[Q', Q+1]$*, with* $Q < Q' \le Q+1$.

Proof. The only part that is yet to be proven is $Q \le Q'$ and $Q' \le 1+Q$. We have

$$Q' - Q = \frac{(1-\alpha_1+\alpha_1 R)(1+\alpha_1)}{(1+\alpha_1+\alpha_1 R)(1+\alpha_1 R)}$$

As $(1-\alpha_1+\alpha_1 R) \le (1+\alpha_1+\alpha_1 R)$ and $1+\alpha_1 \le 1+\alpha_1 R$, we have that $Q'-Q \le 1$. As $R \ge 1$, the numerator is always positive. Hence $0 < Q'-Q \le 1$. With our lower bound on the price of stability of Q', the theorem follows. □

3.4 Specific Reward Sharing Rules

In this section we consider some particularly natural reward sharing rules and show that games with such rules have nice properties. Specifically, while for general reward sharing an (integral) stable matching may not exist, for the reward sharing rules below we show they always exist (although only if there is no social context involved) and how to compute them efficiently. We also give improved bounds on prices of anarchy for these special cases. Specifically, we consider the following sharing rules:

- *Matthew Effect sharing:* In sociology, "Matthew Effect" is a term coined by Robert Merton to describe the phenomenon which says that, when doing similar work, the more famous person tends to get more credit than other less-known collaborators. We model such phenomena for our network by associating brand values λ_u with each node u, and defining the reward that node u gets by getting matched with node v as $r^u_{uv} = \frac{\lambda_u}{\lambda_u+\lambda_v} \cdot r_{uv}$. Thus nodes u and v split the edge reward in the ratio of $\lambda_u : \lambda_v$, and a node with high λ_u value gets a disproportionate amount of reward.

- *Parasite sharing:* This effect is opposite to the Matthew effect in the sense that by collaborating with a renowned person, a less-known person becomes famous, whereas the reputation of the already renowned person does not change significantly from such a collaboration. We model this situation by defining the reward that node u gets by getting matched with node v as $r^u_{uv} = \frac{\lambda_v}{\lambda_u+\lambda_v} r_{uv}$. Thus nodes u and v split the edge reward in the ratio of $\lambda_v : \lambda_u$, in the exactly opposite way to the Matthew Effect sharing.

- *Trust sharing:* Often people collaborate based on not only the quality of a project but also how much they trust each other. We model such a situation by associating a value β_u with each node u, which represents the *trust value* of player u, or how pleasant they are to work with. Each edge (u, v) also has an inherent quality h_{uv}. Then, the reward obtained by node u from partnering with node v is $r^u_{uv} = h_{uv} + \beta_v$.

For the sake of analysis, Matthew Effect sharing and Parasite sharing are the same if we change λ_u of Parasite sharing to $1/\lambda_u$ of Matthew Effect sharing. We will refer to both the models as Matthew Effect sharing from now on.

Existence With friendship utilities, even these intuitive special cases of reward sharing do not guarantee the existence of an integral stable matching; see examples in Section 3.5. Without friendship, however, an integral stable matching exists and can be efficiently computed for Matthew Effect sharing and Trust sharing, unlike in the case of general reward sharing [19].

Theorem 8. *An integral stable matching always exists in stable matching games with Matthew Effect sharing and Trust sharing if $\vec{\alpha} = 0$ (i.e., if there is no friendship). Furthermore, this matching can be found in $O(|V||E|)$ time.*

Proof. Let us define a *preference cycle* as a cycle $(u_1, u_2, \cdots, u_k)$ in the graph G such that $r^{u_i}_{u_i u_{i+1}} \geq r^{u_i}_{u_i u_{i-1}}$ with at least one inequality being strict. Chung [16] defines *odd rings* and proves that if a graph does not contain odd rings, then a stable matching exists. It is straightforward to see that absence of preference cycles implies absence of odd rings. Hence, if a graph has no preference cycles, then a stable matching must exist. Below we prove the stronger statement that such a matching can also be found efficiently.

In brief, we show below that whenever there exist no preference cycles in a graph, we can always find two nodes which prefer getting matched to each other over other nodes. We allow them to get matched to each other and eliminate such matched nodes from the graph. Neither of these two nodes will ever deviate from this matching. Applying the same greedy scheme on the reduced graph will give us a stable matching. Then we will prove that this algorithm produces a stable matching in $O(|V||E|)$ time. Let us now proceed to the details.

Let T_u denote the sets of "best" neighbors of u as follows:

$$T_u = \{v \in N_1(u) : r^u_{uv} \geq r^u_{uw} \ \forall (uw) \in G\} \ . \tag{25}$$

Now we construct a directed graph G_D as follows. For all nodes u, choose a node $v \in T_u$ and draw an edge from u directed to v. Every node in this graph has one outgoing edge, so this graph contains a (directed) cycle. If we find a cycle of length 2 then we have found two nodes which prefer each other the most. If a (directed) cycle $(u_1, u_2, \dots, u_k)$ has length $k > 2$, then we have $r^{u_i}_{u_i u_{i+1}} \geq r^{u_i}_{u_i u_{i-1}}$. Now we cannot have $r^{u_2}_{u_2 u_3} > r^{u_2}_{u_1 u_2}$, otherwise in the original graph G, $(u_1, u_2, \dots, u_k)$ would have constituted a preference cycle. Hence we have $r^{u_2}_{u_1 u_2} = r^{u_2}_{u_2 u_3}$. Thus u_1 and u_3 both are u_2's most preferred nodes. But we also have u_1 prefer u_2 the most as G_D has an edge from u_1 to u_2. Hence u_1 and u_2 is the pair of nodes that prefer each other the most.

Therefore we will always be able to find two nodes in G which prefer each other the most in their preference lists. Match them to each other and they will never have incentive to deviate from this matching. Remove these two nodes and repeat the procedure until no more nodes can be matched. Because no nodes matched in this process will ever deviate, we obtain a stable matching.

It takes $O(|E|)$ time to find each matched pair because for each edge we check if two nodes prefer each other the most. Since the total number of nodes to be matched are $O(|V|)$, we find a stable matching in $O(|V||E|)$ time, as long as there are no preference cycles. All that is left to show is that Matthew effect sharing and Trust sharing do not lead to preference cycles.

Suppose a preference cycle exists in Matthew Effect sharing. Then there exists a cycle $(u_1, u_2, \dots, u_k)$ such that

$$\frac{\lambda_{u_i}}{\lambda_{u_i} + \lambda_{u_{i+1}}} r_{u_i u_{i+1}} \quad \geq \quad \frac{\lambda_{u_i}}{\lambda_{u_i} + \lambda_{u_{i-1}}} r_{u_i u_{i-1}} \tag{26}$$

with at least one inequality being strict. Multiplying all these inequalities and canceling common factors, we reach a contradiction that $1 > 1$. Thus, a preference cycle cannot exist in Matthew Effect sharing.

Suppose a preference cycle exists in Trust sharing. Then there exists a cycle $(u_1, u_2, \dots, u_k)$ such that

$$h_{u_i u_{i+1}} + \beta_{u_{i+1}} \quad \geq \quad h_{u_i u_{i-1}} + \beta_{u_{i-1}} \tag{27}$$

with at least one inequality being strict. Adding all these inequalities and canceling common factors, we reach a contradiction that $0 > 0$. Thus, a preference cycle cannot exist in Trust sharing. □

Price of Anarchy The price of anarchy of Matthew effect sharing can be as high as the guarantee of Theorem 6, with $R = \max_{(uv)} \frac{\lambda_u}{\lambda_v}$. For Trust sharing, however, things are much better:

Theorem 9. *The price of anarchy for (fractional) stable matching games with Trust sharing and friendship utilities is at most* $\max\{2 + 2\alpha_1, 3\}$.

Proof. We first introduce some notation. We denote by M^* an optimum fractional matching and by M a fractional or integral stable matching. Let x^*_{uv} (or x_{uv}) denote the fraction of edge (u, v)

present in M^* (or M). Furthermore,

$$\begin{aligned}
S &= \{e \in E(G) : x^*_e > x_e\} \\
T &= \{e \in E(G) : x^*_e \le x_e\} \\
S_u &= \{e \in E(G) : e \text{ is incident on } u \text{ and } e \in S\} \\
T_u &= \{e \in E(G) : e \text{ is incident on } u \text{ and } e \in T\} \\
E_u &= \text{All edges incident on } u\text{. Note that } E_u = S_u \cup T_u \\
y_u &= 1 - \sum_{(u,v)\in E_u} x_{uv} \\
y^*_u &= 1 - \sum_{(u,v)\in E_u} x^*_{uv} \\
\Delta_u &= \sum_{(u,v)\in T_u} (x_{uv} - x^*_{uv}) + \max\{(y_u - y^*_u), 0\} \\
c^{uv}_u &= (x^*_{uv} - x_{uv})/\Delta_u
\end{aligned}$$

We will show below that the following is true:

$$\sum_{(u,v)\in S} \left(h_{uv} + \frac{\beta_u}{2} + \frac{\beta_v}{2}\right)(x^*_{uv} - x_{uv}) \le \max\{2+2\alpha_1, 3\} \cdot \sum_{(u,w)\in T} \left(h_{uw} + \frac{\beta_u}{2} + \frac{\beta_w}{2}\right)(x_{uw} - x^*_{uw}) \ . \quad (28)$$

This implies that

$$\sum_{(u,v)\in S\cup T} (2h_{uv} + \beta_u + \beta_v)\cdot x^*_{uv} \quad \le \quad \sum_{(u,w)\in S\cup T} \max\{3, 2+2\alpha_1\}\cdot(2h_{uw} + \beta_u + \beta_w)\cdot x_{uw} \ . \quad (29)$$

Now $S \cup T$ covers all the edges in the graph. Also, for trust sharing with friendship utilties we have $q^x_{xy} = (1+\alpha_1)h_{uv} + \alpha_1\beta_u + \beta_v$, thus $q_{xy} = q^x_{xy} + q^y_{xy} = (1+\alpha_1)(2h_{uv} + \beta_u + \beta_v)$. Using this, Eqn (29) implies that the price of anarchy is at most $\max\{2+2\alpha_1, 3\}$. All that remains to show is Eqn (28) which we do next.

To start with, notice that each edge $(u, v) \in S$ can be classified into two categories: a) $(u, v) \in S$ such that $y_u \le y^*_u$ and $y_v \le y^*_v$ AND b) $(u, v) \in S$ such that $y_u > y^*_u$ and $y_v \le y^*_v$. Note that we cannot have both $y_u > y^*_u$ and $y_v > y^*_v$, because then the fraction x_{uv} can be increased by $\min(y_u - y^*_u, y_v - y^*_v)$ by decreasing y_u and y_v by the same quantity. Both u and v would improve their utility in such a case, thus M could not be a (fractional) stable matching.

We will show that whichever category $(u, v) \in S$ belongs to, for one of the endpoints, say u, the following inequality will hold true with $\zeta = \max\{1/2, \alpha_1\}$:

$$\begin{aligned}
\left(h_{uv} + \frac{1}{2}\beta_u + \frac{1}{2}\beta_v\right)(x^*_{uv} - x_{uv}) \quad \le \quad & \sum_{(u,w)\in T_u} \left((1+\alpha_1)h_{uw} + \frac{1}{2}\beta_u + \beta_w\right)(x_{uw} - x^*_{uw})c^{uv}_u \\
& + \sum_{(v,z)\in T_v} \left((1+\alpha_1)h_{vz} + \zeta\beta_v + \zeta\beta_z\right)(x_{vz} - x^*_{vz})c^{uv}_v \ . \quad (30)
\end{aligned}$$

We call u a *witness* node for $(u, v) \in S$. We will observe that adding all the inequalities like Eqn (30) corresponding to each edge $(u, v) \in S$ leads us to Eqn (28), thus proving the theorem in turn.

Now let us see how Eqn (30) can be proved for every edge $(u, v) \in S$. As mentioned before, $(u, v) \in S$ can be classified into two categories and we will prove Eqn (30) for each of them.

1. $(u,v) \in S$ **with** $y_u \le y_u^*$ **and** $y_v \le y_v^*$**:**
   Here we increase x_{uv} to x_{uv}^* by decreasing fraction of each $(u,w) \in T_u$ by $(x_{uw} - x_{uw}^*)c_{uv}^u$ and decreasing fraction of each $(v,z) \in T_v$ by $(x_{vz} - x_{vz}^*)c_{uv}^v$. As M is a stable matching, this does not improve the utility of at least one of the endpoints of (u,v), say u. Call u a witness node for edge (u,v). Since the utility of node u does not improve, we get

$$\begin{aligned} q_{uv}(x_{uv}^* - x_{uv}) \quad &\le \quad \sum_{(u,w)\in T_u} q_{uw}(x_{uw} - x_{uw}^*)c_u^{uv} + \alpha_1 \cdot \sum_{(v,z)\in T_v} r_{vz}^v (x_{vz} - x_{vz}^*)c_v^{uv} \\ &+ \quad \alpha_{uz} \cdot \sum_{(v,z)\in T_v} r_{vz}^z (x_{vz} - x_{vz}^*)c_v^{uv} \ . \end{aligned} \tag{31}$$

   Eqn (31) can be explained as follows: Its left-hand side represents utility gained by u by increasing x_{uv} to x_{uv}^*. The first summation on the right hand side represents the utility lost by u because of decreasing x_{uw} by $(x_{uw} - x_{uw}^*)c_u^{uv}$ for each $(u,w) \in T_u$. The second summation on the right hand side represents the utility lost by u by virtue of friendship with v when we decrease x_{vz} by $(x_{vz} - x_{vz}^*)c_v^{uv}$ for each $(v,z) \in T_v$. The third summation on the right hand side represents the utility lost by u by virtue of friendship with w when we decrease x_{vz} by $(x_{vz} - x_{vz}^*)c_v^{uv}$ for each $(v,z) \in T_v$. Because for trust sharing we have $r_{xy}^x = h_{xy} + \beta_u$ and $q_{xy}^x = (1+\alpha_1)h_{uv} + \alpha_1\beta_u + \beta_v$, Eqn (31) implies that

$$\begin{aligned} ((1+\alpha_1)h_{uv} + \alpha_1\beta_u + \beta_v)(x_{uv}^* - x_{uv}) \quad &\le \quad \sum_{(u,w)\in T_u} ((1+\alpha_1)h_{uw} + \alpha_1\beta_u + \beta_w)(x_{uw} - x_{uw}^*)c_u^{uv} \\ &+ \quad \sum_{(v,z)\in T_v} ((\alpha_1 + \alpha_{uz})h_{vz} + \alpha_{uz}\beta_v + \alpha_1\beta_z)(x_{vz} - x_{vz}^*)c_v^{uv} \ . \end{aligned} \tag{32}$$

   Eqn (32) has similar interpretation as Eqn (31). To simplify Eqn (32), notice that when $y_u \le y_u^*$, calculating Δ_u does not involve $y_u - y_u^*$ giving us $\sum_{(u,w)\in T_u} (x_{uw} - x_{uw}^*)c_u^{uv} = (x_{uv}^* - x_{uv})$. Thus we have

$$(1/2 - \alpha_1)\beta_u(x_{uv}^* - x_{uv}) = (1/2 - \alpha_1)\beta_u \sum_{(u,w)\in T_u} (x_{uw} - x_{uw}^*)c_u^{uv} \ . \tag{33}$$

   Let us add Eqn (33) to Eqn (32) and replace β_v by $\beta_v/2$ on the left-hand side. Additionally, using $\alpha_{uz} \le \alpha_1 \le \zeta$ results into the following equation:

$$\begin{aligned} \left((1+\alpha_1)h_{uv} + \frac{1}{2}\beta_u + \frac{1}{2}\beta_v\right)(x_{uv}^* - x_{uv}) \quad &\le \quad \sum_{(u,w)\in T_u} \left((1+\alpha_1)h_{uw} + \frac{1}{2}\beta_u + \beta_w\right)(x_{uw} - x_{uw}^*)c_u^{uv} \\ &+ \sum_{(v,z)\in T_v} ((1+\alpha_1)h_{vz} + \zeta\beta_v + \zeta\beta_z)(x_{vz} - x_{vz}^*)c_v^{uv} \ . \end{aligned} \tag{34}$$

   Thus we have proved (30) holds for a node u acting as witness for $(u,v) \in S$ such that $y_u \le y_u^*$ and $y_v \le y_v^*$.

2. $(u,v) \in S$ **with** $y_u > y_u^*$ **and** $y_v \le y_v^*$**:**
   Here we find a constant $\epsilon > 0$ such that $\epsilon \cdot (x_{uv}^* - x_{uv}) \le y_u - y_u^*$. Then we increase x_{uv} by $\epsilon \cdot (x_{uv}^* - x_{uv})$ by decreasing y_u by the same amount and by decreasing each $(v,z) \in T_v$ by $\epsilon \cdot (x_{vz} - x_{vz}^*)c_v^{uv}$. By doing this the utility of at least one of the endpoints of (u,v), does not improve because M is a stable matching. This endpoint can be either u or v. We will prove that Eqn (30) holds for each of these cases.

   - Suppose the utility of node u does not improve: Call u a witness node for edge $(u,v) \in S$. Since the utility of node u does not improve, we have

$$((1+\alpha_1)h_{uv} + \alpha_1\beta_u + \beta_v) \cdot \epsilon(x_{uv}^* - x_{uv}) \le \sum_{(v,z)\in T_v} ((\alpha_1 + \alpha_{uz})h_{vz} + \alpha_{uz}\beta_v + \alpha_1\beta_z) \cdot \epsilon(x_{vz} - x_{vz}^*)c_v^{uv} \ . \tag{35}$$

Eqn (35) can be explained on the similar lines of Eqn (31) and (32). The only difference being that the first summation from Eqn (32) is absent from Eqn (35) as fractions x_{uw} do not change for $(u, w) \in T_u$. Canceling ϵ from each side of Eqn (35), we get

$$((1+\alpha_1)h_{uv} + \alpha_1\beta_u + \beta_v) \cdot (x^*_{uv} - x_{uv}) \le \sum_{(v,z)\in T_v} ((\alpha_1 + \alpha_{uz})h_{vz} + \alpha_{uz}\beta_v + \alpha_1\beta_z) \cdot (x_{vz} - x^*_{vz})c_v^{uv} \quad (36)$$

$$\Rightarrow \quad \alpha_1\beta_u(x^*_{uv} - x_{uv}) \le \sum_{(v,z)\in T_v} ((\alpha_1 + \alpha_{uz})h_{vz} + \alpha_{uz}\beta_v + \alpha_1\beta_z) \cdot (x_{vz} - x^*_{vz})c_v^{uv} \ . \quad (37)$$

We multiply Eqn (37) by $(1/2 - \alpha_1)/\alpha_1$ on both sides and add it to Eqn (36) to get the following:

$$\left((1+\alpha_1)h_{uv} + \frac{\beta_u}{2} + \beta_v\right) \cdot (x^*_{uv} - x_{uv}) \le \sum_{(v,z)\in T_v} \left(\frac{\alpha_1 + \alpha_{uz}}{2\alpha_1}h_{vz} + \frac{\alpha_{uz}}{2\alpha_1}\beta_v + \frac{1}{2}\beta_z\right) \cdot (x_{vz} - x^*_{vz})c_v^{uv} \ . \quad (38)$$

Note that to get to Eqn (38), we are performing division by α_1 which requires $\alpha_1 > 0$. However, also notice that if $\alpha_1 = 0$ (and hence all $\alpha_i = 0$) and $y_u > y^*_u$, then node u can only improve its utility by increasing x_{uv} by $\epsilon \cdot (x^*_{uv} - x_{uv})$. Thus the case of $y_u > y^*_u$ and u not improving its utility does not arise.
Now replacing β_v by $\beta_v/2$ and using $\alpha_{uz} \le \alpha_1$ in Eqn (38), we get

$$\begin{aligned}
\left((1+\alpha_1)h_{uv} + \frac{\beta_u}{2} + \frac{\beta_v}{2}\right) \cdot (x^*_{uv} - x_{uv}) &\le \sum_{(v,z)\in T_v} \left(h_{vz} + \frac{1}{2}\beta_v + \frac{1}{2}\beta_z\right) \cdot (x_{vz} - x^*_{vz})c_v^{uv} \\
\Rightarrow \left((1+\alpha_1)h_{uv} + \frac{\beta_u}{2} + \frac{\beta_v}{2}\right) \cdot (x^*_{uv} - x_{uv}) &\le \sum_{(u,w)\in T_u} \left((1+\alpha_1)h_{uw} + \frac{1}{2}\beta_u + \beta_w\right)(x_{uw} - x^*_{uw})c_u^{uv} \\
&\quad + \sum_{(v,z)\in T_v} \left(h_{vz} + \frac{1}{2}\beta_v + \frac{1}{2}\beta_z\right) \cdot (x_{vz} - x^*_{vz})c_v^{uv} \\
\Rightarrow \left((1+\alpha_1)h_{uv} + \frac{1}{2}\beta_u + \frac{1}{2}\beta_v\right)(x^*_{uv} - x_{uv}) &\le \sum_{(u,w)\in T_u} \left((1+\alpha_1)h_{uw} + \frac{1}{2}\beta_u + \beta_w\right)(x_{uw} - x^*_{uw})c_u^{uv} \\
&\quad + \sum_{(v,z)\in T_v} ((1+\alpha_1)h_{vz} + \zeta\beta_v + \zeta\beta_z)(x_{vz} - x^*_{vz})c_v^{uv} \quad (39)
\end{aligned}$$

Thus, Eqn (30) holds in this case too.

- Suppose the utility of node v does not improve. Call node v a witness node for edge (u, v). As the utility of node v does not improve, we get

$$((1+\alpha_1)h_{uv} + \alpha_1\beta_v + \beta_u) \cdot \epsilon(x^*_{uv} - x_{uv}) \le \sum_{(v,z)\in T_v} ((1+\alpha_1)h_{vz} + \alpha_1\beta_v + \beta_z) \cdot \epsilon(x_{vz} - x^*_{vz})c_v^{uv} \ . \quad (40)$$

Eqn (40) can be explained similarly as Eqn (32) and (31). The only differences are that the roles of u and v are reversed and the summation corresponding to the utility lost by v because of decreasing x_{uw} is absent from the right hand side as fractions x_{uw} do not change for $(u, w) \in T_u$ in this case.

To simplify Eqn (40), we cancel ϵ from each side. We also note that as $y_v \le y^*_v$, calculating Δ_v does not involve $y_v - y^*_v$ giving us $\sum_{(v,z)\in T_v}(x_{vz} - x^*_{vz})c_v^{uv} = x^*_{uv} - x_{uv}$. This implies that

$$(1/2 - \alpha_1)\beta_v(x^*_{uv} - x_{uv}) = (1/2 - \alpha_1)\beta_v \sum_{(v,z)\in T_v} (x_{vz} - x^*_{vz})c_v^{uv} \ . \quad (41)$$

Adding Eqn (41) to Eqn (40) and replacing β_u by $\beta_u/2$ on the left hand side, we get

$$\begin{aligned}
\left((1+\alpha_1)h_{uv} + \frac{1}{2}\beta_v + \frac{1}{2}\beta_u\right)(x^*_{uv} - x_{uv}) &\leq \sum_{(v,z)\in T_v}\left((1+\alpha_1)h_{vz} + \frac{1}{2}\beta_v + \beta_z\right)(x_{vz} - x^*_{vz})c^{uv}_v \\
\Rightarrow \quad \left((1+\alpha_1)h_{uv} + \frac{1}{2}\beta_v + \frac{1}{2}\beta_u\right)(x^*_{uv} - x_{uv}) &\leq \sum_{(v,z)\in T_v}\left((1+\alpha_1)h_{vz} + \frac{1}{2}\beta_v + \beta_z\right)(x_{vz} - x^*_{vz})c^{uv}_v \\
&\quad + \sum_{(u,w)\in T_u}((1+\alpha_1)h_{uw} + \zeta\beta_w + \zeta\beta_w)(x_{uw} - x^*_{uw})c^{uv}_u \quad (42)
\end{aligned}$$

Thus, Eqn (30) holds in this case as well.

We showed that for every edge $(u,v) \in S$, for one of its endpoints, say u, the inequality given by Eqn (30) holds true. Now we will show that adding these inequalities leads us to Eqn (28) which in turn proves the theorem as discussed before. Let us look at the coefficients of various terms after we add these inequalities:

- A term $h_{uw}(x_{uw} - x^*_{uw})$ on the right hand side will have coefficient at most

$$(1+\alpha_1)\cdot(\sum_{(u,v)\in S} c^{uv}_u + \sum_{(w,x)\in S} c^{wx}_w) \ ,$$

  which is at most $2(1+\alpha_1)$ using $\sum_{(xy)\in S} c^{xy}_x \leq 1$.

- Let $u = t(u,v)$ denote that u acts as witness node for edge $(u,v) \in S$. Then a term $\beta_u(x_{uw} - x^*_{uw})$ on the right hand side will appear with coefficient

$$\sum_{\substack{(u,v)\in S \\ u=t(u,v)}} \frac{1}{2}\cdot c^{uv}_u + \sum_{\substack{(u,v)\in S \\ u\neq t(u,v)}} \zeta\cdot c^{uv}_u + \sum_{\substack{(w,x)\in S \\ w=t(w,x)}} c^{wx}_w + \sum_{\substack{(w,x)\in S \\ w\neq t(w,x)}} \zeta\cdot c^{wx}_w \ .$$

  When $\alpha_1 \geq 1/2$, we have $\zeta = \alpha_1$, thus the coefficient of $\beta_u(x_{uw} - x^*_{uw})$ becomes

$$\sum_{\substack{(u,v)\in S \\ u=t(u,v)}} \frac{1}{2}\cdot c^{uv}_u + \sum_{\substack{(u,v)\in S \\ u\neq t(u,v)}} \alpha_1\cdot c^{uv}_u + \sum_{\substack{(w,x)\in S \\ w=t(w,x)}} c^{wx}_w + \sum_{\substack{(w,x)\in S \\ w\neq t(w,x)}} \alpha_1\cdot c^{wx}_w \ .$$

  This quantity can be at most $\alpha_1 + 1$.

  On the other hand, when $\alpha_1 < 1/2$, we have $\zeta = 1/2$, thus the coefficient of $\beta_u(x_{uw} - x^*_{uw})$ becomes

$$\sum_{\substack{(u,v)\in S \\ u=t(u,v)}} \frac{1}{2}\cdot c^{uv}_u + \sum_{\substack{(u,v)\in S \\ u\neq t(u,v)}} \frac{1}{2}\cdot c^{uv}_u + \sum_{\substack{(w,x)\in S \\ w=t(w,x)}} c^{wx}_w + \sum_{\substack{(w,x)\in S \\ w\neq t(w,x)}} \frac{1}{2}\cdot c^{wx}_w \ .$$

  This quantity can be at most $3/2$.

Taking into account these coefficients when we sum the inequalities given by Eqn (30) over all $(u,v) \in S$, we get

$$\sum_{(uv)\in S}\left(h_{uv} + \frac{\beta_u}{2} + \frac{\beta_v}{2}\right)(x^*_{uv} - x_{uv}) \leq \max\{2+2\alpha_1, 3\}\cdot\sum_{(uw)\in T}\left(h_{uw} + \frac{\beta_u}{2} + \frac{\beta_w}{2}\right)(x_{uw} - x^*_{uw})$$

and thereby prove Eqn (28) and our claim in turn. □

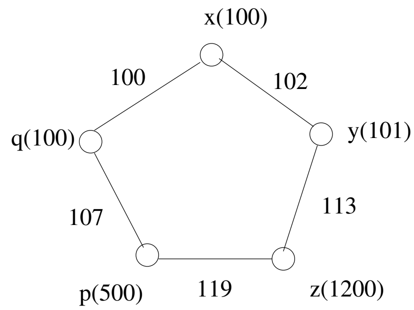

Figure 4: Existence of a stable matching without friendship does not guarantee existence of a stable matching with friendship

3.5 Integral Stable Matchings with Social Context

In Section 2.1 we showed that for equal sharing with friendship utilities, an (integral) stable matching always exists by observing that a stable matching without friendship utilities (i.e. $\vec{\alpha} = \mathbf{0}$) remains a stable matching for friendship utilities.

For unequal reward sharing with friendship, the set of stable matchings for $\vec{\alpha} = \mathbf{0}$ is no longer a subset of the set of stable matchings when we have friendship utilities. Moreover, existence of a stable matching for $\vec{\alpha} = \mathbf{0}$ no more guarantees the existence of a stable matching with friendship utilities. We will give examples below to justify both claims. Finally, we will conclude this section by giving a sufficient condition for the existence of a stable matching for stable matching games with unequal reward sharing and friendship utilities.

The following is an example which has non-overlapping sets of stable matchings with and without friendship: Assign $r^u_{uw} = r^w_{uw} = 1$, $r^u_{uv} = 10/11$, $r^v_{uv} = 100/11$ with $\alpha_1 = 1/2$ and $\alpha_2 = \alpha_3 = \cdots = 0$ in Fig. 2. Without friendship utilities, $\{(u, w)\}$ is the only stable matching as u and w will always want to get matched to each other. However, with friendship utilities we have $q^u_{uv} = \frac{60}{11}$, $q^u_{uw} = \frac{3}{2}, q^v_{uv} = \frac{105}{11}$, $q^v_{uw} = \frac{3}{2}$. Thus, using inequalities (17) and (18) we see that with friendship utilities, the only stable matching is $\{(u, v)\}$ as u will always want to get matched to v. Thus for unequal reward sharing with friendship utilities, the set of stable matchings can be completely nonoverlapping with the set of stable matchings for unequal reward sharing but without friendship utilities.

Next, we give an example with a stable (integral) matching for $\vec{\alpha} = \mathbf{0}$ but no stable (integral) matching with friendship utilities. Consider the Matthew Effect sharing example as shown in Fig. 4. Edge labels indicate edge rewards, values in the brackets beside a node label are the brand values (λ values). By Theorem 8, for $\vec{\alpha} = \mathbf{0}$ a stable matching always exists for Matthew Effect sharing. Let us analyze the example in Fig. 4 with $\alpha_1 = 4/5, \alpha_2 = \alpha_3 = \cdots = 0$. We have

$$\begin{aligned}
q^q_{qx} = 90 &\quad > \quad q^q_{pq} = 89.1667 \\
q^x_{xy} = 91.7493 &\quad > \quad q^x_{qx} = 90 \\
q^y_{yz} = 92.1545 &\quad > \quad q^y_{xy} = 91.8507 \\
q^z_{zp} = 112 &\quad > \quad q^z_{yz} = 111.2455 \\
q^p_{pq} = 103.4333 &\quad > \quad q^p_{zp} = 102.2
\end{aligned}$$

Suppose there exists a stable (integral) matching. In such a matching exactly one node would stay unmatched. Consider the candidate matching $\{(q, x), (z, p)\}$. Now y is unmatched and (x, y) is a blocking pair, because $q^x_{xy} > q^x_{qx}$ and $q^y_{xy} > \alpha_1 r^x_{qx}$. Hence $\{(q, x), (z, p)\}$ is not a stable matching.

Similarly every other matching can be shown to be not stable. Thus, there is no stable matching with friendship utilities, even though with $\vec{\alpha} = 0$ a stable matching exists.

4 Conclusion

We showed that the presence of a social context, such as friendship or altruism, can make a large difference in the existence and the quality of stable matchings, especially if the rewards obtained by neighboring nodes are unequal/unfair. Most of our results can be extended (with minor modifications) to contribution games [8] as well, as they can be considered non-standard fractional versions of stable matching. For details, see our arXiv preprint at [5].

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