

Contribution Games in Networks ^{*}

Elliot Anshelevich [†]

Martin Hoefer [‡]

Abstract

We consider *network contribution games*, where each agent in a network has a budget of effort that he can contribute to different collaborative projects or relationships. Depending on the contribution of the involved agents a relationship will flourish or drown, and to measure the success we use a reward function for each relationship. Every agent is trying to maximize the reward from all relationships that it is involved in. We consider pairwise equilibria of this game, and characterize the existence, computational complexity, and quality of equilibrium based on the types of reward functions involved. When all reward functions are concave, we prove that the price of anarchy is at most 2. For convex functions the same only holds under some special but very natural conditions. Another special case extensively treated are minimum effort games, where the reward of a relationship depends only on the minimum effort of any of the participants. In these games, we can show existence of pairwise equilibrium and a price of anarchy of 2 for concave functions and special classes of games with convex functions. Finally, we show tight bounds for approximate equilibria and convergence of dynamics in these games.

1 Introduction

Understanding the degree to which rational agents will participate in and contribute to joint projects is critical in many areas of society. With the advent of the Internet and the consideration of rationality in the design of multi-agent and peer-to-peer systems, these aspects are becoming of interest to computer scientists and subject to analytical computer science research. Not surprisingly, the study of contribution incentives has been an area of vital research interest in economics and related areas with seminal contributions to the topic over the last decades. A prominent example from experimental economics is the *minimum effort coordination game* [39], in which a number of participants contribute to a joint project, and the outcome depends solely on the minimum contribution of any agent. While the Nash equilibria in this game exhibit a quite simple structure, behavior in laboratory experiments led to sometimes surprising patterns see, e.g., [13, 17, 24, 26, 38] for recent examples. On the analytical side this game was studied, for instance, with respect to logit-response dynamics and stochastic potential in [5].

In this paper we propose and study a simple framework of *network contribution games* for contribution, collaboration, and coordination of actors embedded in networks. The game contains the minimum effort coordination game as a special case and is closely related to many

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[†]Dept. of Computer Science, Rensselaer Polytechnic Institute, Troy, NY. Supported in part by NSF CCF-0914782.

[‡]Dept. of Computer Science, RWTH Aachen University, Germany. mhoefer@cs.rwth-aachen.de. Supported by DFG through UMIC Research Center at RWTH Aachen University and by grant Ho 3831/3-1.

other games from the economics literature. In such a game each player is a vertex in a graph, and the edges represent bilateral relationships that he can engage in. Each player has a budget of effort that he can contribute to different edges. Budgets and contributions are non-negative numbers, and we use them as an abstraction for the different ways and degrees by which actors can contribute to a relationship, e.g., by allocating time, money, and personal energy to maintaining a friendship or a collaboration. Depending on the contribution of the involved actors a relationship will flourish or drown, and to measure the success we use a reward function for each relationship. Finally, each player strives to maximize the total success of all relationships he is involved in.

A major issue that we address in our games is the impact of collaboration. An incentive for collaboration evolves naturally when agents are embedded in (social) networks and engage in relationships. We are interested in the way that a limited collaboration between agents influences properties of equilibria in contribution games like existence, computational complexity, the convergence of natural dynamics, as well as measures of inefficiency. In particular, in addition to unilateral strategy changes we will allow pairs of players to change their strategies in a coordinated manner. States that are resilient against such bilateral deviations are termed *2-strong* [4] or *pairwise equilibria* [31]. This adjustment raises a number of interesting questions. What is the structure of pairwise equilibria, and what are conditions under which they exist? Can we compute pairwise equilibria efficiently or at least efficiently decide their existence? Are there natural improvement dynamics that allow players to reach a pairwise equilibrium (quickly)? What are the prices of anarchy and stability, i.e., the ratios of the social welfare of the best possible state over the worst and best welfare of an equilibrium, respectively? These are the main questions that motivate our study. Before describing our results, we proceed with a formal introduction of the model.

1.1 Network Contribution Games

We consider *network contribution games* as models for the contribution to relationships in networked environments. In our games we are given a simple and undirected graph $G = (V, E)$ with n nodes and m edges. Every node $v \in V$ is a *player*, and every edge $e \in E$ represents a *relationship* (collaboration, friendship, etc.). A player v has a given *budget* $B_v \geq 0$ of the total amount of *effort* that it is able to apply to all of its relationships (i.e., edges incident to v). Budgets are called *uniform* if $B_u = B_v$ for any $u, v \in V$. In this case, unless stated otherwise, we assume that $B_v = 1$ for all $v \in V$ and scale reward functions accordingly.

We denote by E_v the set of edges incident to v . A *strategy* for player v is a function $s_v : E_v \rightarrow \mathbb{R}_{\geq 0}$ that satisfies $\sum_{e=(v,u)} s_v(e) \leq B_v$ and specifies the amount of effort $s_v(e)$ that v puts into relationship $e \in E_v$. A *state* of the game is simply a vector $s = (s_1, \dots, s_n)$. The success of a relationship e is measured by a *reward function* $f_e : \mathbb{R}_{\geq 0}^2 \rightarrow \mathbb{R}$, for which $f_e(x, y) \geq 0$ and non-decreasing in $x, y \geq 0$. The *utility* or *welfare* of a player v is simply the total success of all its relationships, i.e., $w_v(s) = \sum_{e=(v,u)} f_e(s_v(e), s_u(e))$, so both endpoints benefit equally from the undirected edge e . In addition, we will assume that reward functions f_e are symmetric, so $f_e(x, y) = f_e(y, x)$ for all $x, y \geq 0$, and for ease of presentation we will assume they are continuous and differentiable, although most of our results can be obtained without these assumptions.

We are interested in the existence and computational complexity of stable states, their performance in terms of social welfare, and the convergence of natural dynamics to equilibrium. The central concept of stability in strategic games is the (*pure*) *Nash equilibrium*, which is resilient against unilateral deviations, i.e., a state s such that $w_v(s_v, s_{-v}) \geq w_v(s'_v, s_{-v})$ for

each $v \in V$ and all possible strategies s'_v . For the *social welfare* $w(s)$ of a state s we use the natural utilitarian approach and define $w(s) = \sum_{v \in V} w_v(s)$. A *social optimum* s^* is a state with $w(s^*) \geq w(s)$ for every possible state s of the game. Note that we restrict attention to states without randomization and consider only pure Nash equilibria. In particular, the term “Nash equilibrium” will only refer to the pure variant throughout the paper.

In games such as ours, it makes sense to consider multilateral deviations, as well as unilateral ones. Nash equilibria have shortcomings in this context, for instance for a pair of adjacent nodes who would – although being unilaterally unable to increase their utility – benefit from cooperating and increasing the effort jointly. The prediction of Nash equilibrium that such a state is stable is quite unreasonable. In fact, it is easy to show that when considering pure Nash equilibria, the function $\Phi(s) = w(s)/2$ is an exact potential function for our games. This means that s^* is an optimal Nash equilibrium, the price of stability for Nash equilibria is always 1, and iterative better response dynamics converge to an equilibrium. Additionally, for many natural reward functions f_e , the price of anarchy for Nash equilibria remains unbounded.¹

Following the reasoning in, for example, [30,31], we instead consider pairwise equilibrium, and focus on the more interesting case of *bilateral deviations*. An improving bilateral deviation in a state s is a pair of strategies (s'_u, s'_v) such that $w_v(s'_u, s'_v, s_{-u,v}) > w_v(s)$ and $w_u(s'_u, s'_v, s_{-u,v}) > w_u(s)$. A state s is a *pairwise equilibrium* if it is a Nash equilibrium and additionally there are no improving bilateral deviations. Notice that we are actually using a stronger notion of pairwise stability than described in [30], since any pair of players can change their strategies in an arbitrary manner, instead of changing their contributions on just a single edge. In particular, in a state s a coalition $C \subseteq V$ has a *coalitional deviation* s'_C if the reward of every player in C is strictly greater when all players in C switch from strategies s_C to s'_C . s is a *strong equilibrium* if no coalition $C \subseteq V$ has a coalitional deviation. Our notion of pairwise equilibrium is exactly the notion of *2-strong equilibrium* [4], the restriction of strong equilibrium to deviations of coalitions of size at most 2.

We evaluate the performance of stable states using prices of anarchy and stability, respectively. The *price of anarchy (stability)* for pairwise equilibria in a game is the worst-case ratio of $w(s^*)/w(s)$ for the worst (best) pairwise equilibrium s in this game. For a class of games (e.g., with certain convex reward functions) that have pairwise equilibria, the price of anarchy (stability) for pairwise equilibria is simply the worst price of anarchy (stability) for pairwise equilibria of any game in the class. If we consider classes of games, in which existence is not guaranteed, the prices are defined as the worst prices of any game in the class that has pairwise equilibria. Note that unless stated otherwise, the terms price of anarchy and stability refer to pairwise equilibria throughout the paper.

1.2 Results and Contribution

We already observed above that in every game there always exist pure Nash equilibria. In addition, iterative better response dynamics converge to a pure Nash equilibrium, and the price of stability for Nash equilibria is 1. The price of anarchy for Nash equilibria, however, can be arbitrarily large, even for very simple reward functions.

If we allow bilateral deviations, the conditions become much more interesting. Consider the effort $s_v(e)$ expended by player v on an edge $e = (u, v)$. The fact that f_e is monotonic nondecreasing tells us that w_v increases in $s_v(e)$. Depending on the application being considered,

¹Consider, for instance, a path of length 3 with (e_1, e_2, e_3) and $f_{e_1}(x, y) = f_{e_3}(x, y) = \min(x, y)$ and $f_{e_2}(x, y) = M \cdot \min(x, y)$, for some large number M .

however, the utility could possess the property of “diminishing returns”, or on the contrary, could increase at a faster rate as v puts more effort on e . In other words, for a fixed effort amount $s_u(e)$, f_e as a function of $s_v(e)$ could be a concave or a convex function, and we will distinguish the treatment of the framework based on these properties.

	Existence	Price of Anarchy
General convex	Yes (*)	2
General concave	Not always	2
$c_e \cdot (x + y)$	Decision in P	1
Minimum effort convex	Yes (**)	2 (**)
Minimum effort concave	Yes	2
Maximum effort	Yes	2
Approximate Equilibrium	OPT is a 2-apx. Equilibrium	

Table 1: Summary of some of our results for various types of reward functions. For the cases where equilibrium always exists, we also give algorithms to compute it, as well as convergence results. All of our PoA upper bounds are tight. (*) If $\forall e, f_e(x, 0) = 0$, NP-hard otherwise. (**) If budgets are uniform, NP-hard otherwise.

In Section 2 we consider the case of convex reward functions. For a large class \mathcal{C} of convex functions defined below (Definition 2.2) we can show a tight bound for the price of anarchy of 2 (Theorem 2.9). However, for games with functions from \mathcal{C} pairwise equilibria might not exist. In fact, we show that it is NP-hard to decide their existence, even when the edges have simple reward functions of either the form $f_e(x, y) = c_e \cdot (xy)$ or $f_e(x, y) = c_e \cdot (x + y)$ for constants $c_e > 0$ (Theorem 2.7). If, however, *all* functions are of the form $f_e(x, y) = c_e \cdot (xy)$, then existence and efficient computation are guaranteed. We show this existence result for a substantially larger class of functions that may not even be convex, although it includes the class of all convex functions f_e with $f_e(x, 0) = 0$ (Theorem 2.5). Our procedure to construct a pairwise equilibrium in this case actually results in a strong equilibrium, i.e., the derived states are resilient to deviations of every possible subset of players. As the prices of anarchy and stability for pairwise equilibria are exactly 2, they extend to strong equilibria simply by restriction.

As an interesting special case, we prove that if all functions are $f_e(x, y) = c_e \cdot (x + y)$, it is possible to determine efficiently if pairwise equilibria exist and to compute them in polynomial time in the cases they exist (Theorem 2.8).

In Section 3 we consider pairwise equilibria for concave reward functions. In this case, pairwise equilibria may also not exist. Nevertheless, in the cases when they exist, we can show tight bounds of 2 on prices of anarchy and stability (Theorem 3.1).

Sections 4 and 5 treat different special cases of particular interest. In Section 4 we study the important case of minimum effort games with reward functions $f_e(x, y) = h_e(\min(x, y))$. If functions h_e are convex, pairwise equilibria do not necessarily exist, and it is NP-hard to decide the existence for a given game (Theorem 4.5). Perhaps surprisingly, if budgets are uniform, i.e., if

$B_v = B_u$ for all $u, v \in V$, then pairwise equilibria exist for all convex functions h_e (Theorem 4.2), and the prices of anarchy and stability for pairwise equilibria are exactly 2 (Theorem 4.3). If functions h_e are concave, we can always guarantee existence (Theorem 4.8). Our bounds for concave functions in Section 3 imply tight bounds on the prices of anarchy and stability of 2. Most results in this section extend to strong equilibria. In fact, the arguments in all the existence proofs can be adapted to show existence of strong equilibria, and tight bounds on prices of anarchy and stability follow simply by restriction.

In Section 5 we briefly consider maximum effort games with reward functions $f_e(x, y) = h_e(\max(x, y))$. For these games bilateral deviations essentially reduce to unilateral ones. Hence, pairwise equilibria exist, they can be found by iterative better response using unilateral deviations, and the price of stability is 1 (Theorem 5.2). In addition, we can show that the price of anarchy is exactly 2, and this is tight (Theorem 5.3).

Sections 6 to 7 treat additional aspects of pairwise equilibria. In Section 6 we consider approximate equilibria and show that a social optimum s^* is always a 2-approximate equilibrium (Theorem 6.1). In Section 7 we consider sequential and concurrent best response dynamics. We show that for general convex functions and minimum effort games with concave functions the dynamics converge to pairwise equilibria (Theorems 7.2 and 7.4). For the former we can even provide a polynomial upper bound on the convergence times.

Note that almost all of our results on the price of anarchy for pairwise equilibria result in a (tight) bound of 2. This bound of 2 is essentially due to the dyadic nature of relationships, i.e., the fact that edges are incident to at most two players. The case when edges are projects among arbitrary subsets of actors is termed *general contribution game* and treated in Section 8. Here we consider setwise equilibria, which allow deviations by subsets of players that are linked via a joint project. For some classes of such games we show similar results for setwise equilibria as for pairwise equilibria in network contributions games. In particular, we extend the results on existence and price of anarchy for general convex functions and minimum effort games with convex functions. The price of anarchy for setwise equilibria becomes essentially k , where k is the cardinality of the largest project. However, many of the aspects of this general case remain open, and we conclude the paper in Section 9 with this and other interesting avenues for further research.

1.3 Related Work

The model most related to ours is the co-author model [30,31]. The motivation of this model is very similar to ours, although there are many important differences. For example, in the usual co-author model, the nodes cannot choose how to split their effort between their relationships, only which relationship to participate in. Moreover, we consider general reward functions, and as described above, our notion of pairwise stability is stronger than in [30,31].

Our games are potential games with respect to unilateral deviations and can thus be embedded in the framework of congestion games. The social quality of Nash equilibrium in non-splittable atomic congestion games, where the quality is measured by social welfare instead of social cost, has been studied in [35]. Our games allow players to split their effort arbitrarily between incident edges (i.e., they are atomic *splittable* congestion games [37]), and we focus on coalitional equilibrium notions like pairwise stability, not Nash equilibrium. In addition, the reward functions (e.g., in minimum effort games) are much more general and quite different from delay functions usually treated in the congestion game literature [11,18].

In [14], Bramoullé and Kranton consider an extremely general model of network games

designed to model public goods. Nevertheless, our game is not a special case of this model, since in [14] the strategy of a node is simply a level of effort it contributes, not how much effort it contributes *to each relationship*. There are many extensions to this model, e.g., Corbo et al. [20] consider similar models in the context of contributions in peer-to-peer networks. Their work closely connects to the seminal paper on contribution games by Ballester et al. [9], which has prompted numerous similar follow-up studies.

The literature on games played in networks is too diverse to survey here – we will address only the most relevant lines of research. In the last few years, there have been several fascinating papers on network bargaining games (e.g., [16,33]), and in general on games played in networks where every edge represents a two-player game (e.g., [21,22,29]). All these games either require that every node plays the same strategy on all neighboring edges, or leaves the node free to play any strategy on any edge. While every edge in our game can be considered to be a (very simple) two-player game, the strategies/contributions that a node puts on every edge are neither the same nor arbitrarily different: specifically they are constrained by a budget on the total effort that a node can contribute to all neighboring edges in total. To the best of our knowledge, there have been no contributions (other than the ones mentioned below) to the study of games of this type.

Our game bears some resemblance to network formation games where players attempt to maximize different forms of network centrality [2,10,15,25,32,34], although our utility functions and equilibrium structure are very different. Minimum effort coordination games as proposed by van Huyck et al. [39] represent a special case of our general model. They are a vital research topic in experimental economics, see the papers mentioned above and [23] for a recent survey. We study a generalized and networked variant in Section 4. Slightly different adjustments to networks have recently appeared in [3,12]. Our work complements this body of work with provable guarantees on the efficiency of equilibria and the convergence times of dynamics.

Some of the special cases we consider are similar to stable matching [28], and in fact correlated variants of stable matching can be considered an “integral” version of our game. Our results generalize existence and convergence results for correlated stable matching (as, e.g., in [1]), and our price of anarchy results greatly generalize the results of [6].

It is worth mentioning the connection of our reward functions with the “Combinatorial Agency” framework (see, e.g., [7,8]). In this framework, many people work together on one project, and the success of this project depends in a complex (usually probabilistic) manner on whether the people involved choose a high level of effort. It is an interesting open problem to extend our results to the case in which every project of a game is an instance of the combinatorial agency problem.

A related but much more coordinated framework is studied in charity auctions, which can be used to obtain contributions of rational agents for charitable projects. This idea has been first explored by Conitzer and Sandholm [19], and mechanisms for a social network setting are presented by Ghosh and Mahdian [27].

2 Polynomials and Convex Reward Functions

In this section we start by considering a class of reward functions that guarantee a small price of anarchy. We first introduce the notions of a *coordinate-convex* and *coordinate-concave* function.

Definition 2.1. *A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is*

- coordinate-convex if for all of its arguments x_i , we have that $\frac{\partial^2 f}{\partial x_i^2} \geq 0$. A function is strictly coordinate-convex if all these are strict inequalities.
- coordinate-concave if for all of its arguments x_i , we have that $\frac{\partial^2 f}{\partial x_i^2} \leq 0$. A function is strictly coordinate-concave if all these are strict inequalities.

Note that every convex function is coordinate-convex, and similarly every concave function is coordinate-concave. However, coordinate-convexity/concavity is necessary but not sufficient for convexity/concavity. For instance, the function $\log(1 + xy)$ is coordinate-concave, but not concave – indeed, it is convex if $x = y \in [0, 1]$.

Definition 2.2. Class \mathcal{C} consists of all symmetric nondecreasing functions $f : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ that are coordinate-convex. Define class \mathcal{C}' as the subclass of functions from \mathcal{C} that are strictly coordinate-convex. Define class \mathcal{C}_0 as the subclass of functions from \mathcal{C} that satisfy $f(x, 0) = 0$ for all $x \geq 0$.

The class \mathcal{C} is of particular interest to us, because we can show the following result. The proof will appear below in Section 2.2.

Theorem. For the class of network contribution games with reward functions $f_e \in \mathcal{C}$ for all $e \in E$ that have a pairwise equilibrium, the prices of anarchy and stability for pairwise equilibria are exactly 2.

Before we attack the proof, however, let us give some more intuition about functions that belong to \mathcal{C} and the properties of pairwise equilibria in the corresponding games. Consider a polynomial $p(x, y)$ in two variables with non-negative coefficients that is symmetric (i.e., $p(x, y) = p(y, x)$) and non-negative for $x, y \geq 0$. For every such polynomial p we consider all possible extensions to a function $f(x, y) = h(p(x, y))$ with $h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ being nondecreasing and convex. We call the union of all these extensions the class \mathcal{P} . Clearly, every $p(x, y) \in \mathcal{P}$ since $h(x) = x$ is convex. In particular, \mathcal{P} contains a large variety of functions such as xy , $(x + y)^2$, e^{x+y} , $x^3 + y^3 + 2xy$, etc. Observe that $\mathcal{P} \subset \mathcal{C}$, and thus the price of anarchy result for \mathcal{C} will hold for every game with arbitrary functions from \mathcal{P} .

Claim 2.3. It holds that $\mathcal{P} \subset \mathcal{C}$.

Proof. Let $f(x, y) = h(p(x, y))$ be an arbitrary function in \mathcal{P} as described above. f is clearly monotone nondecreasing. $\partial_{xx}p$, $\partial_{yy}p$, and $\partial_{xy}p$ are non-negative, since p has positive coefficients.

$$\partial_{xx}f(x, y) = \partial_{pp}h(p(x, y)) \cdot (\partial_x p(x, y))^2 + \partial_p h(p(x, y)) \cdot \partial_{xx}p(x, y) \geq 0$$

since h is convex. The same holds for the other second partial derivatives. □

2.1 Existence and Computational Complexity of Pairwise Equilibria

While we will show that the price of anarchy is 2 in Section 2.2, this result says nothing about the existence and complexity of computing pairwise equilibria. In fact, even for simple games with reward functions $f_e(x, y) = c_e \cdot (x + y)$ and small constants c_e , pairwise equilibria can be absent.

Example 2.4. In our example there is a triangle graph with nodes u_1 , u_2 , and u_3 , edges $e_1 = (u_1, u_2)$, $e_2 = (u_2, u_3)$, and $e_3 = (u_3, u_1)$, and uniform budgets. Edge e_i has reward function f_i with $f_1(x, y) = f_2(x, y) = 3(x + y)$, and $f_3(x, y) = 2(x + y)$. A pairwise equilibrium must not allow profitable unilateral deviations. Thus, $s_1(e_1) = s_3(e_2) = 1$, because this is obviously a dominant strategy w.r.t. unilateral deviations. Player 2 can assign his budget arbitrarily. This yields $w_1(s) = 3 + 3s_2(e_1)$ and $w_3(s) = 3 + 3s_2(e_2)$. Changing to a state s' where u_1 and u_3 bilaterally deviate by moving all their budget to e_3 yields $w_1(s') = 3s_2(e_1) + 4 > w_1(s)$ and $w_3(s') = 3s_2(e_2) + 4 > w_3(s)$. Hence, no pairwise equilibrium exists.

Although there are games without pairwise equilibria, there is a large class of functions for which we can show existence and an efficient algorithm for computation.

Theorem 2.5. *A pairwise equilibrium always exists and can be computed efficiently when $f_e \in \mathcal{C}_0$ for all $e \in E$.*

Proof. We sort the edges in E in decreasing order by maximum possible reward $c_{u,v} = f_{u,v}(B_u, B_v)$, and let M be the result of a “greedy matching” algorithm for this order. Specifically, we add edges to M in this decreasing order as long as adding the edge still results in a matching. This algorithm can be made to run in $O(m \log m)$ time. We now show that every state s with $s_v(e) = B_v$ iff $e \in M$ is a pairwise equilibrium. The nodes that are not matched in M can distribute their effort arbitrarily. Their payoff remains 0 since $f_e(x, 0) = 0$.

We show the result by contradiction. First, suppose that a node v is willing to deviate unilaterally. Without loss of generality, we can assume this deviation removes effort from an edge of M , and adds all this effort to a single edge $e = (v, u) \notin M$. We can assume this because when forming its best response, v is maximizing the sum of convex functions under a budget constraint (since all reward functions are coordinate-convex). This means that whenever v has an improving unilateral deviation where it adds effort to several edges, it also has an improving unilateral deviation where it adds all this effort to a single edge.

For any edge $e = (v, u) \notin M$ such that u is matched in M , there is no reason for v to add effort to e , since $s_u(e) = 0$ and $f_e(x, 0) = 0$. If u is not matched in M , then by moving x effort from edge $e' = (v, u') \in M$ to e , v will obtain utility at most $f_e(x, B_u) + f_{e'}(B_v - x, B_{u'})$ instead of $f_e(0, B_u) + f_{e'}(B_v, B_{u'})$. This being an improving deviation implies that

$$f_e(x, B_u) - f_e(0, B_u) > f_{e'}(B_v, B_{u'}) - f_{e'}(B_v - x, B_{u'}). \quad (1)$$

Define $e_v(y) = f_e(y, B_u)$ and $e'_v(y) = f_{e'}(y, B_{u'})$. Since e' is chosen before e by the greedy algorithm, it must be that $c_{e'} \geq c_e$, and so $e_v(B_v) \leq e'_v(B_v)$. Since $e_v(0) = e'_v(0) = 0$, there must be some interval of size x on which e'_v increases at least as much as e_v . But since both e_v and e'_v are convex, the interval $[0, x]$ must be the interval of smallest increase, and the interval $[B_v - x, B_v]$ is the interval of largest increase. This implies that

$$e_v(x) - e_v(0) \leq e'_v(B_v) - e'_v(B_v - x) ,$$

a contradiction with Inequality 1. Therefore, we only need to address bilateral deviations.

Suppose that a node v is willing to deviate by switching some x amount of its effort from edge $e' = (v, u') \in M$ to edge $e = (u, v) \notin M$ as part of a bilateral deviation with u . We can assume w.l.o.g. that e' was the first edge of $E_v \cup E_u$ that was added to M , and so $c_{e'} \geq c_e$. For v to be willing to deviate, it must be that Inequality 1 is satisfied. The rest of the argument proceeds as before. \square

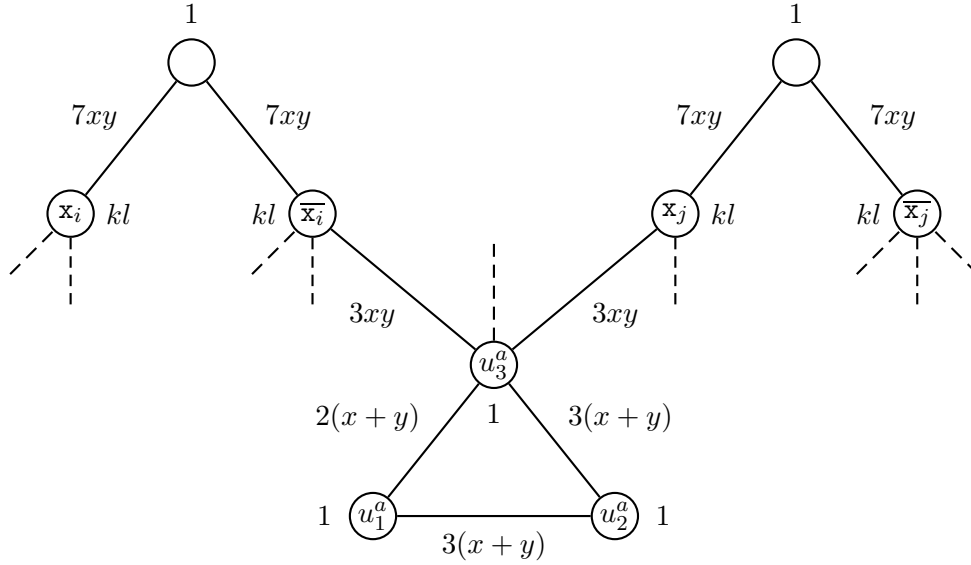


Figure 1: Construction for the NP-hardness proof. Labels inside vertices indicate role of the players, labels at vertices are budgets, and edge labels are reward functions. Vertices in the top layer are decision players with budget 1 corresponding to variables. For each decision player there are two adjacent assignment players (second layer) with budget kl that indicate setting the decision variables. For each clause there is a triangle gadget (third and bottom layer) that by itself has no pairwise equilibrium. Connections between assignment players and triangle gadgets reflect the occurrences of variables in the clauses.

Theorem 2.5 establishes existence and efficient computation of equilibria for many functions from class \mathcal{C} . In particular, it shows existence for all convex functions f_e that are 0-valued when one of its arguments is 0, as well as for many non-convex ones, such as the weighted product function $f_e(x, y) = c_e \cdot (xy)$. In fact, when considering deviations of arbitrary coalitions of players, then it is easy to verify that the player of the coalition incident to the edge with maximum possible reward (of all edges incident to the players in the coalition) does not make a strict improvement in the deviation. Thus, as a corollary we get existence of strong equilibria.

Corollary 2.6. *A strong equilibrium always exists and can be computed efficiently when $f_e \in \mathcal{C}_0$ for all $e \in E$.*

In general, we can show that deciding existence for pairwise equilibria for a given game is NP-hard, even for very simple reward functions from \mathcal{C} such as $f_e(x, y) = c_e \cdot (x + y)$ and $f_e(x, y) = c_e \cdot (xy)$ with constants $c_e > 0$.

Theorem 2.7. *It is NP-hard to decide if a network contribution game admits a pairwise equilibrium even if all functions are either $f_e(x, y) = c_e \cdot (x + y)$ or $f_e(x, y) = c_e \cdot (xy)$.*

Proof. We reduce from 3SAT as follows. We consider a 3SAT formula with k variables and l clauses. For each clause we insert the game of Example 2.4. For each variable we introduce three players as follows. One is a *decision player* that has budget 1. He is connected to two *assignment players*, one true player and one false player. Both the true and the false player

have a budget of $k \cdot l$. The edge between decision and assignment players has $f_e(x, y) = 7xy$. Finally, each assignment player is connected via an edge with $f_e(x, y) = 3xy$ to the player u_3 of every clause triangle, for which the corresponding clause has an occurrence of the corresponding variable in the corresponding form (non-negated/negated).

Suppose the 3SAT instance has a satisfying assignment. We construct a pairwise equilibrium as follows. If the variable in the assignment is set true (false), we make the decision player contribute all his budget to the edge e to the false (true) assignment player. This assignment player will contribute his full budget to e , because $7x$ has steeper slope than $3x$, which is the maximum slope attainable on the edges to the triangle gadgets. It is clear that none of these players has an incentive to deviate (alone or with a neighbor). The remaining set of assignment players A can now contribute their complete budget towards the triangle gadgets. As the assignment is satisfying, every triangle player u_3 of the triangle gadgets has at least one neighboring assignment player in A . We now create a maximum bipartite matching between players in A and the u_3 players of the triangles. We then extend this and connect the remaining (if any) triangle players arbitrarily to assignment players from A . This creates a one-to-many matching of triangle players to players in A , with every triangle player being matched to exactly one player in A , and some players in A possibly unmatched. We set each triangle player to contribute all of his budget towards his edge in the matching. Each assignment player splits his effort evenly between the incident edges in the matching; if the assignment player is unmatched his strategy can be arbitrary. In this matching, each matched assignment player can get up to l matching edges. As the triangle players contribute all their budget to their matching edge, then each edge in the matching yields a reward of $3x$, with x being the contribution of the assignment player. By splitting his budget evenly, the assignment player contributes at least k to each matching edge. Also he receives reward exactly $3kl$, which is the maximum achievable for a player in A (given that the decision player does not contribute to the incident edge). Thus, every matched assignment player in A is stable and will not join a bilateral deviation. Consider a triangle player u_3 . As the assignment player he is matched to contributes at least k (we assume w.l.o.g. $k \geq 3$), the reward function on the matching edge grows at least as quickly as $9y$, i.e., with a larger slope than the maximum slope achievable on the triangle edges. In addition, the reward for u_3 by contributing all budget to the matching edge is at least 9. Note that the maximum payoff that he can obtain by contributing only to triangle edges is 8, and therefore he has no incentive to join other triangle players in a deviation. Note that u_3 could potentially achieve higher revenue by deviating with a different assignment player in A . However, as noted above no matched assignment player has an incentive to deviate jointly with u_3 . Hence, u_3 can only join an unmatched assignment player. This is only a profitable deviation if u_3 currently shares his current assignment player with at least one other triangle player. However, the possibility that u_3 could deviate to such an unmatched assignment player contradicts the fact that we created a maximum matching between assignment and triangle players. Thus, u_3 will also stick to his strategy choice, and has no incentive to participate in bilateral or unilateral deviations. We can stabilize the remaining pairs of triangle players by assigning an effort of 1 towards their joint edge. Finally, the unmatched assignment players v in A are stable since their reward is always 0: no player adjacent to v puts any effort on edges incident to v , and no player adjacent to v is willing to participate in a bilateral deviation due to the arguments above.

Now suppose there is a pairwise equilibrium. Note first that the decision player will always contribute his full budget, and there is always a positive contribution of at least one assignment player towards the decision player edge – otherwise there is a joint deviation that yields higher

reward for both players. In particular, the decision player contributes only to edges, where the maximum contribution of the assignment players is located. As the decision player contributes his full budget, there is at least one incident edge that grows at least as quickly as $3.5x$ in the contribution x of the assignment player. Hence, at least one assignment player will be motivated to remove all contributions from the edges to the triangle players, as these edges grow at most by $3x$ in the contribution x of the assignment player. He will instead invest all of his budget towards the decision player. This implies that every pairwise equilibrium must result in a decision for the variable, i.e., if the (false) true assignment players contributes all of his budget towards the decision player, the variable is set (true) false. If both players do this, the variable can be chosen freely. As there is a stable state, the contributions of the remaining assignment players must stabilize all triangle gadgets. In particular, this means that for each clause triangle there must be at least one neighboring assignment player that does not contribute all of his budget towards his decision player. This implies a satisfying assignment for the 3SAT instance. \square

Finally, let us focus on an interesting special case. The hardness in the previous theorem comes from the interplay of reward functions xy that tend to a clustering of effort and $x + y$ that create cycles. We observed above that if all functions are $c_e \cdot (xy)$, then equilibria exist and can be computed efficiently. Here we show that for the case that $f_e(x, y) = c_e \cdot (x + y)$ for all $e \in E$, we can decide efficiently if a pairwise equilibrium exists. Furthermore, if an equilibrium exists, we can compute it in polynomial time.

Theorem 2.8. *There is an efficient algorithm to decide the existence of a pairwise equilibrium, and to compute one if one exists, when all reward functions are of the form $f_e(x, y) = c_e \cdot (x + y)$ for arbitrary constants $c_e > 0$. Moreover, the price of anarchy is 1 in this case.*

Proof. Let S^* be the set of socially optimum solutions. These are exactly the solutions where every player v puts effort only on edges with maximum c_e . S^* are exactly the solutions that are stable against unilateral deviations, which immediately tells us that if a pairwise equilibrium exists, then the price of anarchy is 1.

Not all solutions in S^* are stable against bilateral deviations, however. Denote $c_v = \max_{e \in E_v} c_e$. Let E_v^* be the set of edges incident to v with value c_v . In any unilaterally stable solution, a node v must put all of its effort on edges in E_v^* . We first show how to determine if a pairwise stable solution exists if the only edges in the graph are $\cup_v E_v^*$.

Consider an edge $e = (u, v)$ such that $e \in E_u^*$ but $e \notin E_v^*$ (call such an edge "Type 1"). Then in any pairwise stable solution, the node u must contribute *all* of its effort to edge e . Otherwise u and v could deviate by v adding some amount $\varepsilon > 0$ to e , and u adding $> \varepsilon(c_v - c_u)/c_u$ to e . This is possible for small enough ε , and would improve the reward for both u (by εc_u) and v (by $> \varepsilon(c_v - c_u) - \varepsilon(c_v - c_u) = 0$). Therefore, for every edge $e = (u, v)$ of this type, we can fix the contributions of node u , since they will be the same in any stable solution. If the same node u has two or more such incident edges e , then by the above argument we immediately know that there does not exist any pairwise equilibrium.

Now consider edges $e = (u, v)$ which are in $E_u^* \cap E_v^*$, which implies that $c_u = c_v$. For any such edge, either $c_u(e) = B_u$ or $c_v(e) = B_v$ in any pairwise equilibrium. If this were not the case, then both u and v could add some ε amount of effort to e and benefit from this deviation by $2\varepsilon c_u$ amount. Consider a connected component consisting of such edges. We can use simple flow or matching arguments to find if there exists an assignment of nodes to edges such that every edge has at least one adjacent node assigned to it. We then set $c_u(e) = B_u$ if node u is assigned to edge e . We also make sure not to assign a node that already used its budget on a Type 1 edge

to *any* edge in this phase. As argued above, if such an assignment does not exist, then there is no pairwise equilibrium. Conversely, any such assignment yields a pairwise equilibrium, since for every edge, at least one of the endpoints of this edge is using all of its effort on this edge. Thus we are able to determine exactly when pairwise equilibria exist on the set of edges $\cup_v E_v^*$. Call the set of such solutions S .

All that is left to check is if one of these solutions is stable with respect to bilateral deviations on edges $e \notin \cup_v E_v^*$. If one solution in S is a pairwise equilibrium in the entire graph, then all of them are, since when moving effort onto an edge e , a node does not care which edge it removes the effort from: all the edges with positive effort have the same slope. To verify that a pairwise equilibrium exists, we simply consider every edge $e = (u, v) \notin \cup_v E_v^*$ with reward function $c_e \cdot (x + y)$, and check if $(c_u - c_e)(c_v - c_e) \geq c_e^2$. We claim that a pairwise equilibrium exists iff this is true for all edges.

Consider a bilateral deviation onto edge e where u contributes ε_1 effort and v contributes ε_2 effort. This would be an improving deviation exactly when $c_e \varepsilon_2 > (c_u - c_e) \varepsilon_1$ and $c_e \varepsilon_1 > (c_v - c_e) \varepsilon_2$. Fix $\varepsilon_1 > 0$ to be some arbitrarily small value; then there exists ε_2 satisfying the above conditions exactly when $(c_u - c_e)/c_e < c_e/(c_v - c_e)$, which is true exactly when $(c_u - c_e)(c_v - c_e) < c_e^2$, as desired. \square

2.2 Price of Anarchy

This section is devoted to proving the following theorem.

Theorem 2.9. *For the class of network contribution games with reward functions $f_e \in \mathcal{C}$ for all $e \in E$ that have a pairwise equilibrium, the prices of anarchy and stability for pairwise equilibria are exactly 2.*

We will refer to an edge $e = (u, v)$ as being *slack* if $B_u > s_u(e) > 0$ and $B_v > s_v(e) > 0$, *half-slack* if $B_u > s_u(e) > 0$ but $s_v(e) \in \{B_v, 0\}$, and *tight* if $s_u(e) \in \{B_u, 0\}$ and $s_v(e) \in \{B_v, 0\}$. We will call a solution *tight* if it has only tight edges.

Claim 2.10. *If all reward functions belong to class \mathcal{C} , then there always exists a tight optimum solution. If all reward functions belong to class \mathcal{C}' , then all optimum solutions are tight.*

Proof. Let s be a solution with maximum social welfare, and let node v be a node that uses non-zero effort on two adjacent edges: $e = (u, v)$ and $e' = (w, v)$. For simplicity, we will denote f_e by f and $f_{e'}$ by g . Furthermore, we denote $s_u(e)$ by α_u , $s_v(e)$ by α_v , $s_v(e')$ by β_v , $s_w(e')$ by β_w . The fact that v switching an ε amount of effort from e to e' or from e' to e does not increase the social welfare means that:

$$f(\alpha_v + \varepsilon, \alpha_u) - f(\alpha_v, \alpha_u) \leq g(\beta_v, \beta_w) - g(\beta_v - \varepsilon, \beta_w) \quad (2)$$

and that

$$g(\beta_v + \varepsilon, \beta_w) - g(\beta_v, \beta_w) \leq f(\alpha_v, \alpha_u) - f(\alpha_v - \varepsilon, \alpha_u). \quad (3)$$

We know from coordinate-convexity that $g(\beta_v, \beta_w) - g(\beta_v - \varepsilon, \beta_w) \leq g(\beta_v + \varepsilon, \beta_w) - g(\beta_v, \beta_w)$. Therefore, we have that

$$f(\alpha_v + \varepsilon, \alpha_u) - f(\alpha_v, \alpha_u) \leq f(\alpha_v, \alpha_u) - f(\alpha_v - \varepsilon, \alpha_u). \quad (4)$$

This is not possible if f is in \mathcal{C}' , giving us a contradiction, and completing the proof for $f \in \mathcal{C}'$. For $f \in \mathcal{C}$, this tells us that v moving its effort from one of these edges to the other will not change the social welfare, and so we can create an optimum solution with one less half-slack edge by setting $\varepsilon = \min(\alpha_v, \beta_v)$. We can continue this process to end up with a tight optimum solution, as desired. \square

Proof of Theorem 2.9. Let s be a pairwise stable solution, and s^* an optimum solution. By Claim 2.10 we can assume that s^* is tight.

Define $w_e(s)$ to be the reward of edge e in s , and $w_e(s^*)$ to be the reward of e in s^* . Recall that for a node v , the utility of v is $w_v(s) = \sum_{e \in E_v} w_e(s)$. Let $e = (u, v)$ be an arbitrary tight edge in s^* . If $s_u^*(e) = B_u$ and $s_v^*(e) = B_v$, then consider the bilateral deviation from s where both u and v put all their effort on edge e . Since s is pairwise stable, there must be some node (wlog node u) such that $w_u(s) \geq w_e(s^*) = f_e(B_u, B_v)$. Make this node u a witness for edge e . If instead $s_u^*(e) = B_u$ and $s_v^*(e) = 0$, then consider the unilateral deviation from s where u puts all its effort on edge e . Since s is stable against unilateral deviation, then $w_u(s) \geq w_e(s^*) = f_e(B_u, 0)$. Make this node u a witness for edge e .

Notice that every node can be a witness for at most one edge, since for a node u to be a witness to edge e , it must be that $s_u^*(e) = B_u$. Therefore, we know that $\sum_v w_v(s) \geq \sum_e w_e(s^*)$. Since the total social welfare in s^* is exactly $2 \sum_e w_e(s^*)$, we know that the price of anarchy for pairwise equilibria is at most 2.

Finally, let us establish tightness of this bound. Consider a path of four nodes with uniform budgets and edges $e_1 = (u, v)$, $e_2 = (v, w)$ and $e_3 = (w, z)$. The reward functions are $f_{e_1}(x, y) = f_{e_3}(x, y) = xy$ and $f_{e_2}(x, y) = (1 + \varepsilon)xy$. v and w achieve their maximum reward by contributing their full budget to e_2 , hence they will apply this strategy in every pairwise equilibrium. This leaves no reward for u and z and gives a total welfare of $2 + 2\varepsilon$. If the players contribute only to e_1 and e_3 , the total welfare is 4. Hence, the price of stability for pairwise equilibria is at least 2, which matches the upper bound on the price of anarchy. \square

For completeness, we also present a result similar to Claim 2.10 for pairwise equilibrium solutions.

Claim 2.11. *If all reward functions belong to class \mathcal{C} , with every reward function f_e having the property that $\frac{\partial^2 f_e}{\partial x \partial y} \geq 0$, then for every pairwise equilibrium, there exists a pairwise equilibrium of the same welfare without slack edges.*

Proof. Let s be a pairwise stable solution, and suppose it contains a slack edge $e = (u, v)$.

This means that nodes u and v have other adjacent edges where they are contributing non-zero effort. Let those edges be $e_1 = (u, w_1)$ and $e_2 = (v, w_2)$. For simplicity, we will denote f_e , by f , f_{e_1} by f_1 , and f_{e_2} by f_2 . Furthermore, we denote $s_v(e)$ by α_v , $s_u(e)$ by α_u , $s_u(e_1)$ by β_u , $s_v(e_2)$ by γ_v and for w_1 and w_2 accordingly.

For any value $\varepsilon \leq \min\{\alpha_u, \beta_u\}$, it must be that u cannot unilaterally deviate by moving ε effort from e to e_1 , or from e_1 to e . Therefore, we know that $f(\alpha_u + \varepsilon, \alpha_v) - f(\alpha_u, \alpha_v) \leq f_1(\beta_u, \beta_{w_1}) - f_1(\beta_u - \varepsilon, \beta_{w_1})$, and that $f_1(\beta_u + \varepsilon, \beta_{w_1}) - f_1(\beta_u, \beta_{w_1}) \leq f(\alpha_u, \alpha_v) - f(\alpha_u - \varepsilon, \alpha_v)$. Since f and f_1 are coordinate-convex, however, we know that $f(\alpha_u, \alpha_v) - f(\alpha_u - \varepsilon, \alpha_v) \leq f(\alpha_u + \varepsilon, \alpha_v) - f(\alpha_u, \alpha_v)$ (and similarly for f_1), which implies that the above inequalities hold with equality. Specifically, it implies that for both f and f_1 , increasing u 's effort by ε causes the same difference in utility as decreasing it by ε . This simply quantifies the fact that for a node

to put effort on more than one edge in a stable solution, it should be indifferent between those two edges.

For any $\varepsilon \leq \min\{\alpha_v, \alpha_u, \beta_u, \gamma_v\}$, consider the pairwise deviation where u and v both move ε amount of effort to e from e_1 and e_2 . Suppose w.l.o.g. that node u is not willing to deviate in this manner because it does not increase its utility. This means that

$$f(\alpha_u + \varepsilon, \alpha_v + \varepsilon) - f(\alpha_u, \alpha_v) \leq f_1(\beta_u, \beta_{w_1}) - f_1(\beta_u - \varepsilon, \beta_{w_1}). \quad (5)$$

By the above argument about u 's unilateral deviations, we know that $f_1(\beta_u, \beta_{w_1}) - f_1(\beta_u - \varepsilon, \beta_{w_1}) = f(\alpha_u + \varepsilon, \alpha_v) - f(\alpha_u, \alpha_v)$, and so we have that $f(\alpha_u + \varepsilon, \alpha_v + \varepsilon) = f(\alpha_u + \varepsilon, \alpha_v)$. Since $\frac{\partial^2 f}{\partial x \partial y} \geq 0$, this also implies that $f(\alpha_u, \alpha_v + \varepsilon) = f(\alpha_u, \alpha_v)$.

We now create solution s' by having node v move ε effort from edge e_2 to e . We will prove below that s' is also a pairwise stable solution of the same welfare as s . However, it has strictly more effort on edge e . We can continue applying the same arguments for edge e until e is no longer a slack edge. This process only decreases the number of slack edges, since we only remove effort from edges that are slack or half-slack, and in the latter case we remove the effort from the "slack" direction, so the edge remains half-slack afterwards.

All that is left to prove is that s' is pairwise stable of the same welfare as s . To see that the welfare is the same, notice that we can use for node v the same arguments for unilateral deviations that we applied to u . Therefore, we know that

$$f(\alpha_u, \alpha_v + \varepsilon) - f(\alpha_u, \alpha_v) = f_2(\gamma_{w_2}, \gamma_v) - f_2(\gamma_{w_2}, \gamma_v - \varepsilon) = 0.$$

Therefore, the reward of all edges in s' , and thus the utility of all nodes, is the same as in s , so the social welfare of both is the same.

We must now prove that s' is pairwise stable. Any possibly improving deviation would have to include one of the edges $e = (u, v)$ or $e_2 = (v, w_2)$, since for all other edges the effort levels are the same in s and s' . First consider deviations (unilateral or bilateral) including node v . Any such deviation would also be a valid deviation in s , since only the strategy of node v has changed. Therefore, all such deviations cannot be improving deviations. Next consider any unilateral deviation by u where u adds some δ effort to edge e . For this to be a strictly improving deviation, it must be that $f(\alpha_u + \delta, \alpha_v + \varepsilon) - f(\alpha_u, \alpha_v + \varepsilon) > 0$. Consider instead a bilateral deviation from s where u plays the new strategy (i.e., adds δ to e), while v deviates by moving ε from e_2 to e . In this deviation, u strictly benefits since it ends in the same configuration as above, and v strictly benefits since it loses $f_2(\gamma_{w_2}, \gamma_v) - f_2(\gamma_{w_2}, \gamma_v - \varepsilon) = 0$ utility and gains $f(\alpha_u + \delta, \alpha_v + \varepsilon) - f(\alpha_u, \alpha_v) > 0$ utility. Therefore, this contradicts s being pairwise stable.

Next consider a deviation from s' where u removes some amount δ from e . For this deviation to be profitable in s' , but not profitable in s , it must be that $f(\alpha_u, \alpha_v + \varepsilon) - f(\alpha_u - \delta, \alpha_v + \varepsilon) < f(\alpha_u, \alpha_v) - f(\alpha_u - \delta, \alpha_v)$. This contradicts the fact that $\frac{\partial^2 f}{\partial x \partial y} \geq 0$.

Finally, consider deviations by node w_2 . If w_2 deviates and removes some amount δ from e_2 , then this is profitable only if $f_2(\gamma_{w_2}, \gamma_v - \varepsilon) - f_2(\gamma_{w_2} - \delta, \gamma_v - \varepsilon) < f_2(\gamma_{w_2}, \gamma_v) - f_2(\gamma_{w_2} - \delta, \gamma_v)$. Recall, however, that $f_2(\gamma_{w_2}, \gamma_v) = f_2(\gamma_{w_2}, \gamma_v - \varepsilon)$, so the above implies that $f_2(\gamma_{w_2} - \delta, \gamma_v - \varepsilon) > f_2(\gamma_{w_2} - \delta, \gamma_v)$, which is impossible since f_2 is nondecreasing in both its arguments. If w_2 adds effort to e_2 in its deviation, then it cannot possibly be more profitable than the same deviation in s , since v is using less effort on e_2 in s' than in s , and the utility of edge e_1 is the same in both. This finishes the proof. \square

Corollary 2.12. *If all reward functions belong to class \mathcal{C}' , then all pairwise equilibria have only tight edges.*

Proof. In the proof of Claim 2.11, we saw that if a node is putting non-zero effort on two edges, then it must be that for some edge e and values x, y , we have that $f_e(x + \varepsilon, y) - f_e(x, y) = f_e(x, y) - f_e(x - \varepsilon, y)$. This is not possible for $f_e \in \mathcal{C}'$, since f_e is strictly convex in each of its arguments. \square

Corollary 2.13. *If all reward functions belong to class \mathcal{C} , then all strict pairwise equilibria (where every player has a unique unilateral best response) have only tight edges.*

Proof. In the proof of Claim 2.11, we saw that if a node is putting non-zero effort on two edges, then its utility does not change by moving some amount of effort from one of these edges to the other. This is not possible in a strict pairwise equilibrium, since then this node would have a deviation that does not change its utility. \square

3 Concave Reward Functions

In this section we consider the case when reward functions $f_e(x, y)$ are concave. It is simple to observe that a pairwise equilibrium may not exist. Consider a triangle graph with three players, uniform budgets, and $f_e(x, y) = \sqrt{xy}$ for all edges. Every player has an incentive to invest his full budget due to monotonic increasing functions. Due to concavity each player will even out the contributions according to the derivatives. Thus, the only candidate for a pairwise equilibrium is when all players put 0.5 on each incident edge. It is, however, easy to see that this state is no pairwise equilibrium. Although we might have no pairwise equilibrium, we obtain the following general result for games with concave rewards that have a pairwise equilibrium.

Theorem 3.1. *For the class of network contribution games with concave reward functions for all $e \in E$ that have a pairwise equilibrium, the price of anarchy for pairwise equilibria is at most 2.*

Proof. Consider a social optimum s^* , and the effort $s_v^*(e)$ used by node v on edge e in this solution. Let s be a pairwise equilibrium, with $s_v(e)$ the effort used by v on edge e in s . For an edge $e = (u, v)$, let $w_e(s) = f_e(s_u(e), s_v(e))$ be its reward in s , and $w_e(s^*) = f_e(s_u^*(e), s_v^*(e))$ be its reward in s^* . We will now attempt to charge $w(s^*)$ to $w(s)$.

For any node v , define O^v to be the set of edges incident to v where v contributes strictly more in s^* than in s , i.e., where $s_v^*(e) > s_v(e)$. Similarly, define S^v to be the set of edges e where $s_v^*(e) \leq s_v(e)$.

Let O be the set of edges e with strictly higher reward in s^* than in s ($w_e(s^*) > w_e(s)$), and S be the rest of the edges in the graph, with reward in s at least as high as in s^* . Furthermore, define $O_1 = \{e = (u, v) | e \in O^u \cap S^v \cap O\}$ and $O_2 = \{e = (u, v) | e \in O^u \cap O^v \cap O\}$. In other words, O_2 is the set of edges with higher reward in s^* where both players contribute more in s^* than in s , and O_1 is the set of edges where only one player does so. Similarly, define $S_1 = \{e = (u, v) | e \in O^u \cap S^v \cap S\}$ and $S_2 = \{e = (u, v) | e \in S^u \cap S^v \cap S\}$. Since reward functions are monotone, every edge must appear in exactly one of O_1, O_2, S_1 , or S_2 .

In the following proof, we will first show that any edge $e = (u, v)$ in O can be assigned to one of its endpoints (say u) such that u would never gain in deviating from s by removing effort from edges in S^u and making its contribution to e equal to $s_u^*(e)$, even if v did the same. This

means that the utility node u loses from setting its contribution to $s_u^*(e)$ instead of $s_u(e)$ on all edges of S^u is at least as much as the difference in utility in s^* versus in s on all the edges of O assigned to u . We then sum up these inequalities, which lets us bound the reward on edges where s^* is better than s by the reward on edges where s is better than s^* . We now proceed with the proof as described.

Let $e = (u, v)$ be an arbitrary edge of O , so $w_e(s^*) > w_e(s)$. Since f_e is nondecreasing, this implies that $s_v(e) < s_v^*(e)$ or $s_u(e) < s_u^*(e)$, i.e., at least one of u or v has strictly lower effort on e in s than in s^* . Consider the deviation from s to another state s' where u and v increase their contributions to e to the same level as in s^* , i.e., a state s' yields $s'_u(e) = \max\{s_u(e), s_u^*(e)\}$ and $s'_v(e) = \max\{s_v(e), s_v^*(e)\}$. This may be either a bilateral or a unilateral deviation, depending on whether one of $s'_u(e) = s_u(e)$ or $s'_v(e) = s_v(e)$ holds. Note that there is actually an entire set of such states s' , as we did not specify from where players u and v potentially remove effort to be able to achieve the increase. Observe, however, that no other player changes his strategy, i.e., $s'_{-u,v} = s_{-u,v}$. Since s is an equilibrium, it must be that for at least one of u or v the deviation to *every possible* such state s' is unprofitable. Without loss of generality, say that this player is u , so $w_u(s') \leq w_u(s)$ for every state s' , and we say that we *assign* edge e to node u . Note that this implies that $\delta_u(e) = s_u^*(e) - s_u(e) > 0$, i.e., $e \in O^u$, since otherwise all edges incident to u would have the same reward in every s' as in s , except for the edge e which would have reward $w_e(s^*)$ in s' , strictly greater than $w_e(s)$.

Since $f_e(s'_u(e), s'_v(e)) \geq w_e(s^*)$, then there is an increase in w_u due to e of at least $w_e(s^*) - w_e(s)$. However, as every deviation to a state s' is unprofitable for u , it must be that removing $\delta_u(e)$ effort in *any arbitrary way* from other edges incident to u and adding it to e would not increase u 's utility w_u . Therefore, we know that, in particular, removing $\delta_u(e)$ effort from edges S^u decreases the reward of those edges by at least $w_e(s^*) - w_e(s)$. Denote by $\chi_u(\delta_u(e))$ this amount, i.e., $\chi_u(\delta)$ is the minimum amount that w_u would decrease if in state s player u removed any δ amount of effort from edges in S^u .

We have now proven that for any $e = (u, v) \in O$, we can assign it to one of its endpoints (say u), such that $\chi_u(\delta_u(e)) \geq w_e(s^*) - w_e(s)$. We can now sum these inequalities for every edge $e \in O$. Consider the sum of just the inequalities corresponding to the edges assigned to a fixed node v (call this set of edges $A(v)$). Then we have that

$$\sum_{e \in A(v)} [w_e(s^*) - w_e(s)] \leq \sum_{e \in A(v)} \chi_v(\delta_v(e)) .$$

How does $\chi_v(\delta_1) + \chi_v(\delta_2)$ compare to $\chi_v(\delta_1 + \delta_2)$? Since all the functions f_e are concave, it is easy to see that removing $\delta_1 + \delta_2$ effort from edges S_v will decrease the reward of these edges by at least as much as the sum of $\chi_v(\delta_1)$ and $\chi_v(\delta_2)$. Therefore, we know that

$$\sum_{e \in A(v)} \chi_v(\delta_v(e)) \leq \chi_v \left(\sum_{e \in A(v)} \delta_v(e) \right) .$$

Since $\delta_v(e)$ is the extra effort of v on edge e in s^* compared to s , the sum of $\delta_v(e)$ for the edges O^v equals $\Delta = \sum_{e \in S^v} s_v(e) - s_v^*(e)$. Thus, $\chi_v(\Delta)$ is at most the utility lost by v if, starting at state s , v would set its contribution to $s_v^*(e)$ instead of $s_v(e)$ on all edges of S^v . For an edge $e \in S_2$, this is at most $w_e(s) - w_e(s^*)$, since even after lowering v 's contribution to $s_v^*(e)$, the reward of this edge is at least $w_e(s^*)$. For an edge $e \in S_1$ or $e \in O_1$, this is still at most $w_e(s)$.

Noticing that an edge of S^v cannot be in O_2 , we now have that

$$\chi_v(\Delta) \leq \sum_{e \in S^v \cap S_2} [w_e(s) - w_e(s^*)] + \sum_{e \in S^v \cap (S_1 \cup O_1)} w_e(s) .$$

Putting this all together, we obtain that

$$\sum_{e \in A(v)} [w_e(s^*) - w_e(s)] \leq \sum_{e \in S^v \cap S_2} [w_e(s) - w_e(s^*)] + \sum_{e \in S^v \cap (S_1 \cup O_1)} w_e(s) .$$

Summing up these inequalities for all nodes v , we obtain a way to bound the reward on edges where s^* is better than s by the reward on edges where s is better than s^* . Since the same edge $e = (u, v)$ could be in both S^u and S^v , it may be used in the above sum twice. Notice, however, that any edge in S_1 or O_1 will only appear in this sum once, since it will belong to S^v of exactly one node. Thus, we obtain that

$$\sum_{e \in O} [w_e(s^*) - w_e(s)] \leq 2 \sum_{e \in S_2} [w_e(s) - w_e(s^*)] + \sum_{e \in S_1 \cup O_1} w_e(s) .$$

Adding in the edges of S_1 , and recalling that all edges are in exactly one of O_1 , O_2 , S_1 , or S_2 , gives us the desired bound:

$$w(s^*) \leq 2 \sum_{e \in S_2} w_e(s) + \sum_{e \in S_1 \cup O_1} w_e(s) + \sum_{e \in O} w_e(s) + \sum_{e \in S_1} w_e(s) \leq 2w(s) .$$

□

Looking carefully at the proof of the previous theorem yields the following result (c.f. Definition 2.1).

Corollary 3.2. *For the class of network contribution games with coordinate-concave reward functions for all $e \in E$ that have a pairwise equilibrium, the price of anarchy for pairwise equilibria is at most 2.*

4 Minimum Effort Games

In this section we consider the interesting case (studied for example in [5, 17, 24, 26, 38, 39]) when all reward functions are of the form $f_e(x, y) = h_e(\min(x, y))$. In other words, the reward of an edge depends only on the minimum effort of its two endpoints. In our treatment we again distinguish between the case of increasing marginal returns (convex functions h_e) and diminishing marginal returns (concave functions h_e). Note that in this case bilateral deviations are in many ways essential to make the game meaningful, as there is almost always an infinite number of Nash equilibria.² In addition, we can assume w.l.o.g. that in every pairwise equilibrium s there is a unique value s_e for each $e = (u, v) \in E$ such that $s_v(e) = s_u(e) = s_e$. The same can be assumed for optima s^* .

We begin by showing a simple yet elegant proof based on linear programming duality, that shows a price of anarchy of 2 when all functions $h_e(x) = c_e \cdot x$ are linear with slope $c_e > 0$. We include this proof to highlight that duality is also used in Theorem 4.3 for convex functions and uniform budgets.

²In particular, due to monotonic increasing functions any state in which for each edge the contributions of incident players are the same is a Nash equilibrium.

Theorem 4.1. *The prices of anarchy and stability for pairwise equilibria in games with all functions of the form $f_e(x, y) = c_e \cdot \min(x, y)$ are exactly 2.*

Proof. We use linear programming duality to obtain the result. Consider an arbitrary pairwise equilibrium s and an optimum s^* . Note that the problem of finding s^* can be formulated as the following linear program, with variables x_e representing the minimum contribution to edge e :

$$\begin{aligned} \text{Max} \quad & \sum_{e \in E} 2c_e x_e \\ \text{s.t.} \quad & \sum_{e: e=(u,v)} x_e \leq B_u \quad \text{for all } u \in V \\ & x_e \geq 0 \quad . \end{aligned} \tag{6}$$

The LP-dual of this program is

$$\begin{aligned} \text{Min} \quad & \sum_{u \in V} B_u y_u \\ \text{s.t.} \quad & y_u + y_v \geq 2c_e \quad \text{for all } e \in E \\ & y_u \geq 0 \quad . \end{aligned} \tag{7}$$

Now consider the pairwise equilibrium s and a candidate dual solution y composed of

$$y_u = \sum_{e: e=(u,v)} \frac{c_e s_e}{B_u}.$$

If a player contributes all of his budget in s , this is the average payoff per unit of effort. Note that $\{s_e\}$ is a feasible primal, and $\sum_e 2c_e s_e = \sum_u y_u B_u$, but $\{y_u\}$ is not a feasible dual solution. Now suppose that for an edge e both incident players u and v have $y_u, y_v < c_e$. Then both incident players can either move effort from an edge with below-average payoff to e , or invest some of their remaining budget on e . This increases both their payoffs and contradicts that s is stable. Thus, for every edge e there is a player u with $y_u \geq c_e$. Thus, by setting $y'_u = 2y_u$ we obtain a feasible dual solution with profit of twice the profit of s . The upper bound follows by standard duality arguments. It is straightforward to derive a tight lower bound on the price of stability using a path of length 3 and functions $h_e(x) = x$ and $h_e(x) = (1 + \varepsilon)x$ in a similar fashion as presented in Theorem 2.9 previously. \square

4.1 Convex Functions in Minimum Effort Games

In this section we consider reward functions $f_e(x, y) = h_e(\min(x, y))$ with convex functions $h_e(x)$. This case bears some similarities with our treatment of the class \mathcal{C} in Section 2. In fact, we can show existence of pairwise equilibria in games with uniform budgets. We call an equilibrium s *integral* if $s_e \in \{0, 1\}$ for all $e \in E$.

Theorem 4.2. *A pairwise equilibrium always exists in games with uniform budgets and $f_e(x, y) = h_e(\min(x, y))$ when all h_e are convex. If all h_e are strictly convex, all pairwise equilibria are integral.*

Proof. We first show how to construct a pairwise equilibrium. The proof is basically again an adaptation of the “greedy matching” argument that was used to show existence for general convex functions in Theorem 2.8. In the beginning all players are asleep. We iteratively wake

up the pair of sleeping players that achieves the highest revenue on a joint edge and assign them to contribute their total budget towards this edge. The algorithm stops when there is no pair of incident sleeping players.

Suppose for contradiction that the resulting assignment is not a pairwise equilibrium. First consider a bilateral deviation, where a pair of players can profit from re-assigning some budget to an edge e' . By our algorithm at least one of the players incident to e' is awake. Consider the incident player u that was woken up earlier. If it is profitable for him to remove some portion x of effort from an edge e to e' , this implies

$$h_e(1) < h_e(1-x) + h_{e'}(x)$$

However, our choices imply $h_e(1) \geq h_{e'}(1)$. Convexity yields $h_{e'}(x) \leq xh_{e'}(1)$ and $h_e(1-x) \leq (1-x)h_e(1)$ and results in a contradiction

$$\begin{aligned} h_e(1) &< h_e(1-x) + h_{e'}(x) \\ &\leq (1-x)h_e(1) + xh_{e'}(1) \\ &\leq (1-x)h_e(1) + xh_e(1) \\ &= h_e(1) . \end{aligned}$$

This implies that the algorithm computes a stable state with respect to bilateral deviations. As for unilateral deviations, no player would ever add any effort to an edge where the other endpoint is putting in zero effort. However, if a player u unilaterally re-assigns some x budget to an edge $e' = (u, v)$ from edge e with v still being asleep at the end of the algorithm, then this implies that $h_e(1) < h_e(1-x) + h_{e'}(x)$ and that $h_e(1) \geq h_{e'}(1)$. This gives a contradiction by the same argument as above.

If all functions are strictly convex, then $h_e(x) < xh_e(1)$ for all $x \in (0, 1)$. In this case we show that every stable state s is integral, i.e., we have $s_e \in \{0, 1\}$. Suppose to the contrary that there is an equilibrium s with $s_e \in (0, 1)$ for $e = (u, v)$. Let e be an edge with the largest value $h_e(1)$ such that $s_e \in (0, 1)$. For player u , let e_i with $i = 1, \dots$ be other incident edges of u such that $s_{e_i} \in (0, 1)$. Then, because of strict convexity, we have

$$h_e(s_e) + \sum_i h_{e_i}(s_{e_i}) < s_e h_e(1) + \sum_i s_{e_i} h_{e_i}(1) \leq h_e(1) .$$

This means u has an incentive to move all of his effort to e if v does the same. By the same argument, v also has an incentive to move all its effort to e . Thus, the bilateral deviation of u and v moving their effort to e is an improving deviation for both u and v , so we have a contradiction to s being stable. \square

Theorem 4.3. *The prices of anarchy and stability for pairwise equilibria in network contribution games are exactly 2 when all reward functions $f_e(x, y) = h_e(\min(x, y))$ with convex h_e , and budgets are uniform.*

Proof. Consider a stable solution s and an optimum solution s^* . For a vertex u we consider the profit and denote this by $y_u = \sum_{e:e=(u,v)} f_e(s_e)$. For every edge $e = (u, v)$, consider the case when both players invest the full effort. Due to convexity $s_e^* \cdot f_e(1) \geq f_e(s_e^*)$. Suppose for both players $y_u, y_v < f_e(1)$. Then there is a profitable switch by allocating all effort to e . This implies that $\max\{y_u, y_v\} \geq f_e(1)$ and thus,

$$(y_u + y_v) \cdot s_e^* \geq s_e^* f_e(1) \geq f_e(s_e^*) .$$

Thus, we can bound

$$\sum_{e \in E} (y_u + y_v) \cdot s_e^* \geq \sum_{e \in E} f_e(s_e^*) = w(s^*)/2 .$$

On the other hand

$$\sum_{e \in E} (y_u + y_v) \cdot s_e^* = \sum_{u \in V} \sum_{e: e=(u,v)} y_u \cdot s_e^* \leq \sum_{u \in V} y_u = 2 \sum_{e \in E} f_e(s_e) = w(s) .$$

Hence, $w(s) \geq w(s^*)/2$ and the price of anarchy is 2.

Note that this is tight for functions f that are arbitrarily convex. The example is a path of length 3 similar to Theorem 2.9 and Theorem 4.1. We use $f_e(1) = 1 + \varepsilon$ for the inner edge and $f_e(1) = 1$ for the outer edges, and the price of stability becomes arbitrarily close to 2. \square

For the case of arbitrary budgets and convex functions, however, we can again find an example that does not allow a pairwise equilibrium.

Example 4.4. Our example game consists of a path of length 3. We denote the vertices along this path with u, v, w, z . All players have budget 2, except for player z that has budget 1. The profit functions are $h_{u,v}(x) = 2x^2$, $h_{v,w}(x) = 5x$, and $h_{w,z}(x) = 6x$. Observe that this game allows no pairwise equilibrium: If $2 \geq s_{v,w} > 1$, then player w has an incentive to increase the effort towards z . If $1 \geq s_{v,w} > 0$, then player v has an incentive to increase effort towards u . If $s_{v,w} = 0$, both v and w can jointly increase their profits by contributing 2 on (v, w) .

Using this example we can construct games in which deciding existence of pairwise equilibria is hard.

Theorem 4.5. *It is NP-hard to decide if a network contribution game admits a pairwise equilibrium if budgets are arbitrary and all functions are $f_e(x, y) = h_e(\min(x, y))$ with convex h_e .*

Proof. We reduce from 3SAT and use a similar reduction to the one given in Theorem 2.7. An instance of 3SAT is given by k variables and l clauses. For each clause we construct a simple game of Example 4.4 that has no stable state. For each variable we introduce three players as follows. One is a *decision player* that has budget $k \cdot l$. He is connected to two *assignment players*, one true player and one false player. Both these players have also a budget of $k \cdot l$. The edge between decision and assignment players has $h_e(x) = 10x^2$. Finally, each assignment player is connected via an edge with $h_e(x) = 7x$ to the node z of every clause path, for which the corresponding clause has an occurrence of the corresponding variable in the corresponding form (non-negated/negated). Note that the connecting player z is the only player with budget 1 in the clause path.

Suppose the 3SAT instance has a satisfying assignment. We construct a stable state as follows. If the variable is set true (false), we make the decision player contribute all his budget to the edge e to the false (true) assignment player. Both assignment player and decision player are motivated to contribute their full budget to e , because $10(kl)^2$ is the maximum profit that they will ever be able to obtain. Clearly, none of these players has an incentive to deviate (alone or with a neighbor). The remaining set of assignment players A can now contribute their complete budget towards the clause gadgets. As the assignment is satisfying, every node z of the clause gadgets has at least one neighboring assignment player in A . We create a maximum bipartite matching of clause players z to players in A and match the remaining clause players z (if any) to players from A arbitrarily. Each clause player z contributes all of his budget towards

his edge in this one-to-many matching. Each assignment player splits his effort evenly between the incident edges in the matching. Note that the players z from the clause gadgets now receive profit 7, which is the maximum achievable. Thus, they have no incentive to deviate. Hence, no player in A has a profitable unilateral or a possible bilateral deviation. Finally, we obtain a stable state in the clause gadgets by assigning all players v and w to contribute 2 to (v, w) .

Now suppose there is a stable state. Note first that the decision player and one incident assignment player can and will obtain their maximum profit by contributing their full budgets towards a joint edge – otherwise there is a joint deviation that yields higher profit for both players. Hence, this assignment player will not contribute to edges to the clause gadget players. This implies a decision for the variable, i.e., if the (false) true assignment players contributes all of his budget towards the decision player, the variable is set (true) false. As there is a stable state, the contributions of the remaining assignment players must stabilize all clause gadgets. In particular, this means that for each clause triangle there must be at least one neighboring assignment player that does not contribute towards his decision player. This implies that the assignment decisions made by the decision players must be satisfying for the 3SAT instance. \square

The construction of Example 4.4 and the previous proof can be extended to show hardness for games with uniform budgets in which functions are either concave or convex.

Corollary 4.6. *In games with uniform budgets and functions $f_e(x, y) = h_e(\min(x, y))$ with monotonic increasing h_e it is NP-hard to determine if a pairwise equilibrium exists.*

Proof. We use the same approach as in the previous proof, however, we assign each player a budget of $B_u = k \cdot l$. For each of the players u, v, w , and z in a clause gadget we introduce players u', v', w' and z' . u' is only connected to u , v' only to v , and similar for w' and z' . The edges (u, u') , (v, v') and (w, w') have profit function $h_e(x) = 10 \cdot (kl - 2)^{1.5} \cdot \sqrt{x}$ for $x \leq kl - 2$ and $h_e(x) = 10(kl - 2)^2$ otherwise. Similarly, we use $h_e(x) = 10 \cdot (kl - 1)^{1.5} \cdot \sqrt{x}$ for $x \leq kl - 1$ and $h_e(x) = 10(kl - 1)^2$ otherwise for (z, z') . It is easy to observe that in every pairwise equilibrium players u and u' will contribute $kl - 2$ towards their joint edge. This holds accordingly for every other pair of players (v, v') , (w, w') and (z, z') . The remaining budgets of the players are the budgets used in Example 4.4 above and lead to the same arguments in the above outlined reduction. \square

Finally, we observe that the existence result in Theorem 4.2 extends to strong equilibria. In particular, whenever we consider a deviation from a coalition of players, the reward of players incident to the highest reward edge do not strictly improve by the deviation. In addition, the prices of anarchy and stability are 2 because our lower bound examples continue to hold for strong equilibria, while the upper bounds follow by restriction.

Corollary 4.7. *A strong equilibrium always exists in games with uniform budgets and $f_e(x, y) = h_e(\min(x, y))$ when all h_e are convex. If all h_e are strictly convex, all strong equilibria are integral. The prices of anarchy and stability for strong equilibria in these games are exactly 2.*

4.2 Concave Functions in Minimum Effort Games

In this section we consider the case of diminishing returns, i.e., when all h_e are concave functions. Note that in this case the function $f_e = h_e(\min(x, y))$ is coordinate-concave. Therefore, the results from Section 3 show that the price of anarchy is at most 2. However, for general coordinate-concave functions it is not possible to establish the existence of pairwise equilibria,

which we do for concave h_e below. In fact, if the functions h_e are strictly concave, we can show that the equilibrium is unique.

Theorem 4.8. *A pairwise equilibrium always exists in games with $f_e(x, y) = h_e(\min(x, y))$ when all h_e are continuous, piecewise differentiable, and concave. It is possible to compute pairwise equilibria efficiently within any desired precision. Moreover, if all h_e are strictly concave, then this equilibrium is unique.*

Proof. First, notice that we can assume without loss of generality that for every edge $e = (u, v)$, the function h_e is constant for values greater than $\min(B_u, B_v)$. This is because it will never be able to reach those values in any solution.

We create a pairwise equilibrium in an iterative manner. For any solution and set of nodes S , define $BR_v(S)$ as the set of best responses for node v if it can control the strategies of nodes S . We begin by computing $BR_v(V)$ independently for each player v (V is the set of all nodes). In particular, this simulates that v is the player that always creates the minimum of every edge, and we pick s_v such that it maximizes $\sum_{e=(u,v)} h_e(s_v(e))$. This is a concave maximization problem (or equivalently a convex minimization problem), for which it is possible to find a solution by standard methods in time polynomial in the size of G , the encoding of the budgets B_v and the number of bits of precision desired for representing the solution. For background on efficient algorithms for convex minimization see, e.g., [36].

Let $h_e^+(x)$ be the derivative of $h_e(x)$ in the positive direction, and $h_e^-(x)$ be the derivative of $h_e(x)$ in the negative direction. We have the property that for s_v calculated as above, for every edge e with $s_v(e) > 0$ it holds that $h_e^-(s_v(e)) \geq h_{e'}^+(s_v(e'))$ for every edge e' incident to v . Define h'_v as the minimum value of $h_e^-(s_v(e))$ for all edges e incident to v with $s_v(e) > 0$.

Our algorithm proceeds as follows. At the start all players are asleep, and in each iteration we pick one player to wake up. Let S_i denote the set of sleeping players in iteration i , and $A_i = V - S_i$ the set of awake players; in the beginning $S_1 = V$. We will call edges with both endpoints asleep *sleeping* edges, and all other edges *awake* edges.

In each iteration i , we pick one player to wake up, and fix its contributions on all of its adjacent edges. In particular, we choose a node $v \in S_i$ with the currently highest derivative value h'_v (see below for tie-breaking rule). We set v 's contribution to an edge $e = (u, v)$ to $s_v(e)$, where $s_v \in BR_v(S_i)$. Define $\overline{BR}_v(S_i)$ as the set of best responses in $BR_v(S_i)$ for which $s_v(e) = s_u(e)$ for all awake edges $e = (u, v)$. For $s_v \in \overline{BR}_v(S_i)$ player v exactly matches the contributions of the awake nodes A_i on all awake edges between v and A_i . By Lemma 4.10 below, $\overline{BR}_v(S_i)$ is non-empty, and our algorithm sets the contributions of v to $s_v \in \overline{BR}_v(S_i)$. Moreover, we set the contribution of other sleeping players $u \in S_i$ to be $s_u(e) = s_v(e)$ on the sleeping edges, so we assume u fully matches v 's contribution on edge e . By Lemma 4.10, u will not change its contributions on these edges when it is woken up. Thus, in the final solution output by the algorithm v will receive exactly the reward of $\overline{BR}_v(S_i)$. Now that node v is awake, we compute $\overline{BR}_u(S_i - \{v\})$ for all sleeping u , as well as new values h'_u and iterate. Note that values h'_u in later iterations are defined as the minimum derivative values on all the *sleeping* edges neighboring u , not on all edges. To summarize, each iteration i of the algorithm proceeds as follows:

- For every $u \in S_i$, compute $s_u \in \overline{BR}_u(S_i)$.
- For every $u \in S_i$, set h'_u to be the minimum value of $h_e^-(s_u(e))$ for all *sleeping* edges e incident to u with $s_u(e) > 0$.

- Choose a node v with maximum h'_v (using tie-breaking rule below), fix v 's strategy to be s_v , and set $S_{i+1} = S_i - \{v\}$.

To fully specify the algorithm, we need to define a tie-breaking rule for choosing a node to wake up when there are several nodes with equal values h'_v . Let $s_v \in \overline{BR}_v(S_i)$ that we compute. Our goal is that for every edge $e = (u, v)$ with $h'_u = h'_v$, we choose node u such that $s_u(e) \leq s_v(e)$. We claim that we can always find a node u such that this is true with respect to all its neighbors. Suppose a node u has two edges $e = (u, v)$ and $e' = (u, w)$ with $h'_u = h'_v = h'_w$ and $s_u(e) > s_v(e)$ but $s_u(e') < s_w(e')$. Lemma 4.9 below implies that the functions on (u, v) and (u, w) are linear in this range. Specifically, Lemma 4.9 implies that $h_{e'}^+(s_u(e')) = h_e^-(s_u(e))$ because $h_e^-(s_u(e)) = h'_u = h'_w = h_{e'}^-(s_w(e')) \leq h_{e'}^+(s_u(e'))$, with the inequality being true because $h_{e'}$ is concave. Hence, u can move some amount of effort from e to e' and still form a best response. Continuing in this manner, we can find another best response in $\overline{BR}_u(S_i)$ for u such that u has contributions that are either more than both its neighbors, or less than both its neighbors. This implies that there exists $u \in S_i$ with $s_u \in \overline{BR}_u(S_i)$ such that $s_u(e) \leq s_v(e)$ for all neighbors v , and therefore our tie-breaking is possible.

Lemma 4.9. *Consider two nodes u and v and an edge $e = (u, v)$, and let $s_u \in \overline{BR}_u(S_i)$ and $s_v \in \overline{BR}_v(S_i)$ be the best responses computed in our algorithm. Suppose that $s_v(e) > s_u(e)$. Then it must be that either $h'_u > h'_v$, or $h'_u = h_e^-(s_v(e)) = h'_v$.*

Proof. If edge e is the edge which achieves the minimum value h'_u , then we are done, since then $h'_u = h_e^-(s_u(e)) \geq h_e^-(s_v(e)) \geq h'_v$. Therefore, we can assume that another edge $e' = (u, w)$ with $s_u(e') > 0$ achieves this value, so $h'_u = h_{e'}^-(s_u(e'))$.

The fact that we cannot increase u 's reward by assigning more effort to edge e means that $h_{e'}^-(s_u(e')) \geq h_e^+(s_u(e))$. Since h_e is concave, we know that $h_e^+(s_u(e)) \geq h_e^-(s_v(e))$, which is at least h'_v by its definition. This proves that $h'_u \geq h'_v$. If this is a strict inequality, then we are done. The only possible way that $h'_u = h'_v$ is if $h'_u = h_{e'}^-(s_u(e')) = h_e^-(s_v(e)) = h'_v$, as desired. \square

First we will prove that our algorithm forms a feasible solution, i.e., that the budget constraints are never violated. To do this, we must show that when the i 'th node v is woken up and sets its contribution $s_v(e)$ on a newly awake edge $e = (u, v)$, the other sleeping player u must have enough available budget to match $s_v(e)$. In $s_u \in \overline{BR}_u(S_i)$ that our algorithm computes, let \overline{B}_u be the available budget of node u , that is,

$$\overline{B}_u = B_u - \sum_{e=(u,w), w \in A_i} s_w(e) ,$$

the budget minus requested contributions on awake edges. This is the maximum amount that node u could assign to e .

For contradiction, assume that $s_v(e) > \overline{B}_u$, so our assignment is infeasible. Then it must be that $h_e^-(s_u(e)) \geq h_e^-(\overline{B}_u) \geq h_e^-(s_v(e))$, since h_e is concave. By definition of h'_v , we know that $h_e^-(s_v(e)) \geq h'_v$, and so $h_e^-(s_u(e)) \geq h'_v$. Now let $e' = (u, w)$ be the edge that achieves the value h'_u , i.e., $h_{e'}^-(s_u(e')) = h'_u$. If $h_{e'}^-(s_u(e')) < h'_v$, then $h_{e'}^-(s_u(e')) < h_e^-(\overline{B}_u) \leq h_e^+(s_u(e))$, so s_u cannot be a best response, since u could earn more reward by switching some amount of effort from e' to e . Therefore, we know that $h'_u \geq h'_v$. If this is a strict inequality, then we have a contradiction, since u would have been woken up before v . Therefore, it must be that $h'_u = h'_v$. But this contradicts our tie-breaking rule – we would choose u before v because it puts less

effort onto edge e in our choice from $\overline{BR}_u(S_i)$ than v does in $\overline{BR}_v(S_i)$. Therefore, our algorithm creates a feasible solution.

Lemma 4.10. *For every node u and all S_i until node u is woken up, there is a best response in $BR_u(S_i)$ that exactly matches the contributions of the awake nodes A_i . In other words, $\overline{BR}_u(S_i)$ is non-empty.*

Proof. We prove this by induction on i ; this is trivially true for S_1 . Suppose this is true for S_{i-1} , and let v be the node that is woken up in the i 'th iteration, with an existing edge $e = (u, v)$, so that $S_i = S_{i-1} - \{v\}$. Let $s_u \in \overline{BR}_u(S_{i-1})$ be u 's best response which exists by the inductive hypothesis. First, we claim that $s_u(e) \geq s_v(e)$. To see this, notice that if $s_u(e) < s_v(e)$, then by Lemma 4.9, we know that $h'_u \geq h'_v$. If this is a strict inequality, then we immediately get a contradiction, since we picked v to wake up because it had the highest h'_v value. If $h'_u = h'_v$, this contradicts our tie-breaking rule, since u would be woken up first for contributing less to edge e .

Consider the computation of $BR_u(S_{i-1})$ from u 's point of view. u is deciding how to allocate its budget B_u among incident edges, in order to maximize its reward. By putting x effort onto an edge $e' = (u, w)$ with $w \in S_{i-1}$, u will obtain $h_{e'}(x)$ reward, since u can control the strategy of w , and so will make it match the contribution of u on edge e' . If instead $w \in A_{i-1}$, then by putting x effort onto e' , u will only obtain $h_{e'}(\min(x, s_w(e')))$ reward, since the strategy of w is already fixed, and u cannot change it. Then, $BR_u(S_{i-1})$ is simply the set of budget allocations of u that maximizes the sum of the above reward functions. Now consider the computation of $BR_u(S_i)$ and compare it to $BR_u(S_{i-1})$. The only difference is that u cannot control the node v when computing $BR_u(S_i)$, i.e., by putting x effort onto edge e , node u will only obtain $h_e(\min(x, s_v(e)))$ reward, instead of $h_e(x)$.

If $s_u(e) = s_v(e)$, then $s_u \in BR_u(S_i)$ as well as in $BR_u(S_{i-1})$, since the computations of $BR_u(S_i)$ and $BR_u(S_{i-1})$ only differ in the reward function of edge e , and u cannot gain any utility by putting more than $s_u(e)$ effort onto edge e in $BR_u(S_i)$. s_u matches all the contributions of nodes in A_i (including v), and so $\overline{BR}_u(S_i)$ is non-empty.

Suppose instead that $s_u(e) > s_v(e)$. Now, let s'_u be a strategy of u created from s_u as follows. Remove effort from edge e by setting $s'_u(e) = s_v(e)$, and add $s_u(e) - s_v(e)$ effort to the other edges of u in the optimum way to maximize u 's utility in $BR_u(S_i)$. It is easy to see that this is a best response in $BR_u(S_i)$, since a best response in $BR_u(S_i)$ is simply obtained by repeatedly adding effort to the edges with highest derivative. Moreover, s'_u matches the contributions on all edges to A_i , so once again we know that $\overline{BR}_u(S_i)$ is non-empty. \square

Re-number the nodes v_1, v_2, \dots, v_n in the order that we wake them. We need to prove that the algorithm computes a pairwise equilibrium. By Lemma 4.10, we know that all the contributions in the final solution are symmetric, and that node v_i gets exactly the reward $\overline{BR}_{v_i}(S_i)$ in the final solution.

To prove that the above algorithm computes a pairwise equilibrium, we show by induction on i that node v_i will never have incentive to deviate, either unilaterally or bilaterally. This is clearly true for v_1 , since it obtains the maximum possible reward that it could have in *any* solution, which proves the base case. We now assume that this is true for all nodes earlier than v_i , and prove it for v_i as well. It is clear that v_i would not deviate unilaterally, since it is getting the reward of $BR_{v_i}(S_i)$. This is at least as good as any best response when it cannot control the strategies of any nodes except itself. By the inductive hypothesis, v_i would not deviate bilaterally with a node v_j such that $j < i$. v_i would also not deviate bilaterally with a node v_j

such that $j > i$, since when forming $\overline{BR}_{v_i}(S_i)$ node v_i can set the strategy of node v_j . So in $\overline{BR}_{v_i}(S_i)$ node v_i achieves a reward better than any deviation possible with nodes from S_i . This completes the proof that our algorithm always finds a pairwise equilibrium.

Now we will consider the case when all h_e are *strictly* concave, and prove that there is a unique pairwise equilibrium. Consider the algorithm described above. It is greatly simplified for this case: since all h_e are strictly concave, then $BR_v(S_i)$ consists of only a single strategy, and by Lemma 4.10, this strategy is also in $\overline{BR}_v(S_i)$. We claim that when this algorithm assigns a strategy s_v to a node v , then v must have this strategy in *every* pairwise equilibrium. We will prove this by induction, so suppose this is true for all nodes earlier than $v = v_i$, but there is some pairwise equilibrium s' where v does not use the strategy $s_v \in \overline{BR}_v(S_i)$. Since $s'_v \neq s_v$, then there must be some edge $e = (v, u)$ such that $s'_v(e) < s_v(e)$. If u is a node considered earlier than v , then by the inductive hypothesis, we know that $s_u(e) = s'_u(e)$. s_v is the unique best response of v if it were able to control the strategies of nodes in S_i . This means that the gain that v could obtain by moving some small amount of effort to edge e is greater than the loss that it would obtain from removing effort from any edge to a node of S_i , and so v would have a unilateral deviation in s' . If instead $u \in S_i$, then the only way that it would not benefit v to move some effort onto e is if $s'_u(e) = s'_v(e)$. Since v was chosen by the algorithm before u , we know that it would always benefit u to move some effort onto edge e in this case, since the derivative it would encounter there is higher than u encounters on any other edge. Thus, there exists a bilateral deviation where both u and v move some effort onto edge e . \square

For the case of strong equilibria, we observe that the arguments for existence can be adapted, while the upper bounds of 2 on the price of anarchy translate by restriction. In particular, consider a pairwise equilibrium as described in the proof of Theorem 4.8. Resilience to coalitional deviations can be established in exactly the same way as above, i.e., the player from the coalition that was the first to be woken up has no incentive to deviate.

Corollary 4.11. *A strong equilibrium always exists in games with $f_e(x, y) = h_e(\min(x, y))$ when all h_e are continuous, piecewise differentiable, and concave. It is possible to compute strong equilibria efficiently within any desired precision. Moreover, if all h_e are strictly concave, then this equilibrium is unique.*

5 Maximum Effort Games

In this section we briefly consider $f_e(x, y) = h_e(\max(x, y))$ for arbitrary monotonic increasing functions. Our results rely on the following structural observation.

Lemma 5.1. *If there is a bilateral deviation that is strictly profitable for both players, then there is at least one player that has a profitable unilateral deviation.*

Proof. Suppose the bilateral deviation decreases the maximum effort on the joint edge. In this case, both players must receive more profit from other edges. This increase, however, can obviously also be realized by each player himself.

Suppose the bilateral deviation increases the maximum effort on the joint edge. Then the player setting the maximum effort on the edge can obviously also do the corresponding strategy switch by himself, which yields the same outcome for him. \square

It follows that a stable solution always exists, because the absence of unilateral deviations implies that the state is also a pairwise equilibrium. Furthermore, the total profit of all players

is a potential function of the game with respect to unilateral better responses.³ This implies that the social optimum is a stable state and the price of stability is 1.

Theorem 5.2. *A pairwise equilibrium always exists in games with $f_e(x, y) = h_e(\max(x, y))$ and arbitrary monotonic increasing functions h_e . The price of stability for pairwise equilibria is 1.*

We can also easily derive a tight result on the price of anarchy for arbitrary functions.

Theorem 5.3. *The price of anarchy for pairwise equilibria in network contribution games with $f_e(x, y) = h_e(\max(x, y))$ and arbitrary monotonic increasing functions h_e is at most 2. This bound is tight for arbitrary convex functions.*

Proof. For an upper bound on the social welfare of the social optimum s^* consider each player u and suppose that he optimizes his effort independently. This yields a reward f_u . Clearly, $w(s^*) \leq 2 \sum_u f_u$. To see this, notice that in s^* , we can assume that every edge has contribution from only one direction. Let E_u^* be the edges to which u contributes in s^* . In this case, u 's reward from these edges is at most f_u . The reward of the other nodes because of these edges is also at most f_u . Therefore, in total $w(s^*) \leq 2 \sum_u f_u$.

On the other hand, in any pairwise equilibrium s player u will not accept less profit than f_u , because by a unilateral deviation he can always achieve (at least) the maxima used to optimize f_u . Thus, $w_u(s) \geq f_u$, and we have that

$$w(s^*) \leq 2 \sum_{u \in V} f_u \leq 2 \sum_{u \in V} w_u(s) = 2w(s) \quad .$$

Tightness follows from the following simple example. The graph is a path with four nodes u_i , for $i = 1, \dots, 4$. The interior players have budget 1, the leaf players have budget 0. The edges $e_1 = (u_1, u_2)$ and $e_2 = (u_2, u_3)$ have an arbitrary convex functions $h(x)$, the remaining edge e_3 has $h_{e_3}(x) = \varepsilon h(x)$, for an arbitrarily small $\varepsilon > 0$. A pairwise equilibrium s evolves when player u_2 spends his effort on e_2 and u_3 on e_3 . This yields a total profit of $(2 + 2\varepsilon)h(1)$. The optimum evolves if u_2 contributes on e_1 and u_3 on e_2 with total profit of $4h(1)$. \square

6 Approximate Equilibrium

We showed above several classes of functions for which pairwise equilibrium exists, and the price of anarchy is small. If we consider *approximate* equilibria, however, the following theorem says that this is always the case. By an α -approximate equilibrium, we will mean a solution where nodes may gain utility by deviating (either unilaterally or bilaterally), but they will not gain more than a factor of α utility because of this deviation.

Theorem 6.1. *In network contribution games an optimum solution s^* is a 2-approximate equilibrium for any class of nonnegative reward functions.*

Proof. First, notice that s^* is always stable against unilateral deviations. This is because when a node v changes the effort it allocates to its adjacent edges unilaterally, then the only nodes

³Note that the social welfare is not a potential function for bilateral deviations. Consider a path of length 3 with $h_e(x) = 2x$ on the outer edges and $h_e(x) = 3x$ on the inner edge. The inner players have budget 1, the outer players budget 0. If both inner players contribute to the outer edges, their utility is 2. If they both move all their effort to the inner edge, their utility becomes 3. Note, however, that the social welfare decreases from 8 to 6.

affected are neighbors of v . If C is the change in node v 's reward because of its unilateral deviation, then the total change in social welfare is exactly $2C$. Therefore, no node can improve their reward in s^* using unilateral deviations.

Now consider bilateral deviations, and assume for contradiction that nodes u and v have a bilateral deviation by adding some amounts δ_u and δ_v to edge $e = (u, v)$, which increases their rewards by more than a factor of 2. Let $z_u^* = w_u^* - f_e(s_u^*(e), s_v^*(e))$ and $z_v^* = w_v^* - f_e(s_u^*(e), s_v^*(e))$ be the rewards of u and v in s^* not counting edge e . We denote by z_u and z_v the same rewards after u and v deviate by adding effort to e , and therefore possibly taking effort away from other adjacent edges. In other words, the reward of u before the deviation is $w_u^*(s^*) = z_u^* + f_e(s_u^*, s_v^*)$, and after the deviation it is $z_u + f_e(s_u^* + \delta_u, s_v^* + \delta_v)$. Note that this change cannot increase $w(s)$ over $w(s^*)$, therefore, we know that

$$2z_u + 2z_v + 2f_e(s_u^* + \delta_u, s_v^* + \delta_v) \leq 2z_u^* + 2z_v^* + 2f_e(s_u^*, s_v^*) . \quad (8)$$

On the other hand, since both u and v must improve their reward by more than a factor of 2, we know that

$$z_u + f_e(s_u^* + \delta_u, s_v^* + \delta_v) > 2z_u^* + 2f_e(s_u^*, s_v^*) ,$$

and

$$z_v + f_e(s_u^* + \delta_u, s_v^* + \delta_v) > 2z_v^* + 2f_e(s_u^*, s_v^*) .$$

Adding the last two inequalities together, we obtain that

$$z_u + z_v + 2f_e(s_u^* + \delta_u, s_v^* + \delta_v) > 2z_u^* + 2z_v^* + 4f_e(s_u^*, s_v^*) \quad (9)$$

$$\geq 2z_u + 2z_v + 2f_e(s_u^* + \delta_u, s_v^* + \delta_v) + 2f_e(s_u^*, s_v^*) \quad (10)$$

which implies that $z_u + z_v + 2f_e(s_u^*, s_v^*) < 0$, a contradiction. \square

7 Convergence

In this section we consider the convergence of round-based improvement dynamics to pairwise equilibrium. Perhaps the most prominent variant is best response, in which we deterministically and sequentially pick one particular player or a pair of adjacent players in each round and allow them to play a specific unilateral or bilateral deviation. While convergence of such dynamics is desirable, a drawback is that convergence could rely on the specific deterministic sequence of deviations. Here we will consider less demanding processes that allow players or pairs of players to be chosen at random to make deviations, and we even allow concurrent deviations of more than one player or pair.

We consider random best response, where we randomly pick either a single player or one pair of adjacent players in each round and allow them to play a unilateral or bilateral deviation. In each round, we make this choice uniformly at random, i.e, a specific pair (u, v) of players gets the possibility to make a bilateral deviation with probability $1/(n + m)$. In concurrent best response, each player decides independently whether he wants to deviate unilaterally or picks a neighbor for a bilateral deviation. Obviously, a bilateral deviation can be played if and only if both players decide to pick each other. Hence, in a given round a player v decides to play a unilateral deviation with probability $p_v = 1/(\deg_v + 1)$, where \deg_v is the degree of v . A pair (u, v) of players makes a bilateral deviation with probability $p_u \cdot p_v$. Note that in both dynamics, in expectation, after a polynomial number of rounds each single player or pair of players gets the chance to play a unilateral or bilateral deviation.

The name “best response” in our dynamics needs some more explanation for bilateral deviations, because for a pair of players (u, v) a particular joint deviation (s'_u, s'_v) might result in the best reward for u but not for v . In fact, there might be no joint deviation that is simultaneously optimal for both players. In this case the players should agree on one of the Pareto-optimal alternatives.

In this section we consider special kinds of dynamics, which resolve this issue in an intuitive way. The intuition is that if two players decide to play a bilateral deviation, then these strategies should also be unilateral best responses. We assume that players do not pick bilateral deviations, in which they would change the strategies unilaterally. More formally, we capture this intuition by the following definition.

Definition 7.1. *A bilateral best response for a pair (u, v) of players in a state s is a pair (s'_u, s'_v) of strategies that is*

- *a profitable bilateral deviation, i.e., $w_u(s'_u, s'_v, s_{-u,v}) > w_u(s)$, and $w_v(s'_u, s'_v, s_{-u,v}) > w_v(s)$, and*
- *a pair of mutual best responses, i.e., $w_u(s'_u, s'_v, s_{-u,v}) \geq w_u(s''_u, s'_v, s_{-u,v})$ for every strategy s''_u of player u , and similarly for v .*

Note that, in principle, there might be states that allow a bilateral deviation, but there exists no bilateral best response. The set of states resilient to unilateral and bilateral best responses is a superset of pairwise equilibria. Hence, it might not even be obvious that dynamics using only bilateral best responses converge to pairwise equilibria. Our results below, however, show that the latter is true in many of the games for which we showed existence of pairwise equilibria above.

General Convex Functions For games with strictly coordinate-convex functions, the concept of bilateral best response reduces to a simple choice rule. In this case, a unilateral best response of every player places the entire player budget on a single edge. This implies that there is no bilateral best response where players split their efforts. Thus, bilateral best responses come in three different forms, in which the players allocate their efforts towards their joint edge (1) both, (2) only one of them, or (3) none of them.

Consider two incident players u and v in state s connected by edge e . To compute a bilateral best response from one of the forms mentioned above, we proceed in two phases. In the first phase, we try forms (2) and (3) and remove all contributions from e . Then player u independently picks a unilateral best response under the assumption that $s_v(e) = 0$. Note that in case of equal reward a player always prefers to put the effort on e , because this might attract the other player to put effort on e as well and, by convexity, increase their own reward even further. Similarly, we do this for player v . This yields a pair of “virtual” best responses under the condition that the other player does not contribute to e . Now we have to check whether this is a bilateral best response. In particular, if only one of the players puts effort on e , by convexity it might become a unilateral best response for the other player to put his effort on e as well. If this is the case, the computed state is not a pair of mutual best responses, thus the most profitable candidate for a bilateral best response is of form (1).

If this is not the case, then by convexity one player is not willing to contribute to e at all. Hence, his virtual best response is a unilateral best response even though the other player contributes to e . For the other player, this means that the assumption made for the virtual

best response are satisfied, hence, we have found a set of mutual best responses of the form (2) or (3). However, in this case, this set might not be a bilateral best response because the players do not improve over their current reward. We thus also check, whether a state of form (1) is a better set of mutual best responses. Hence, we consider the state of form (1) and the resulting reward for each player. Each reward must be at least as large as that from the virtual best responses, because otherwise the state does not represent a set of mutual best responses. If this is the case, we accept $s'_u(e) = B_u$ and $s'_v(e) = B_v$ as our candidate for the bilateral best response. Otherwise, we use the pair of virtual best responses.

Note that our algorithm computes in each case the most profitable candidate for a bilateral best response, and always finds a bilateral best response if one exists. This can be verified directly for each of the cases, in which there is a bilateral best response of forms (1), (2), or (3).

Theorem 7.2. *Random and concurrent best response dynamics converge to a pairwise equilibrium in a polynomial number of rounds when all reward functions are strictly coordinate-convex and $f_e(0, x) = 0$ for all $e \in E$ and $x \geq 0$.*

Proof. Let us consider the edges in classes of their $c_{u,v}$ (c.f. proof of Theorem 2.5). In particular, our analysis proceeds in phases. In phase 1, we restrict our attention to the first class of edges with the highest $c_{u,v}$ and the subgraph induced by these edges. Consider one such edge e and suppose both players contribute their complete budgets to e . They are never again willing to participate in a bilateral deviation (not only a bilateral best response), because by strict convexity they achieve the maximum possible revenue. We will call such players *stabilized*. Consider a first class edge $e = (u, v)$ where strategies $s_u(e) < B_u$ and $s_v(e) = B_v$. In this case, strict convexity, $f_e(x, y) = 0$ when $xy = 0$, and maximality of $c_{u,v}$ imply that $s'_u(e) = B_u$ is a unilateral best response - independently of the current s_u . If both players have $s_u(e) < B_u$ and $s_v(e) < B_v$, then the same argument implies that both players allocating their full budget is a bilateral best response, again independently of what the current strategies of the players are. Note that each bilateral best response of two destabilized players enlarges the set of stabilized players. Phase 1 ends when there are no adjacent destabilized players with respect to first class edges, and this obviously takes only an expected number of time steps that is polynomial in n .

After phase 1 has ended, we know that the stabilized players are never going to change their strategy again. Hence, for the purpose of our analysis, we drop the edges between stabilized players from consideration. The same can be done for all edges e incident to exactly one stabilized player v , by artificially reducing v 's budget to $B_v = 0$ and noting that $f_e(B_u, 0) = 0$. If there are remaining destabilized players, phase 2 begins, and we consider only the remaining players and the edges among them. In this graph, we again consider only the subgraph induced by edges with highest $c_{u,v}$. Again, we have the property that any pair of players contributing their full budget to such an edge is stabilized. Additionally, the same arguments show that for destabilized players there are always unilateral and/or bilateral best responses that result in investing the full budget, irrespective of the current strategy. Hence, after expected time polynomial in n , phase 2 ends and expands the set of stabilized players by at least 2.

Repeated application of this argument shows that after expected time polynomial in n either all players are stabilized or the remaining subgraph of destabilized players is empty. In this case, a pairwise equilibrium is reached. In particular, using unilateral and bilateral best responses suffices to stabilize all but an independent set of players. It is easy to observe that stabilized players have no profitable unilateral or bilateral deviations. Possibly remaining destabilized players in the end are only adjacent to stabilized players and therefore have no profitable unilat-

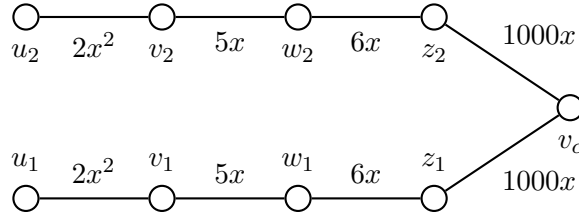


Figure 2: A minimum effort game with convex reward functions and uniform budgets $B_v = 2$. There is a starting state such that no sequence of bilateral best responses can reach a stable state.

eral or bilateral deviations. Thus, our dynamics converge to a pairwise equilibrium in expected polynomial time. This proves the theorem. \square

Minimum Effort Games and Convex Functions In this section we show that there are games with infinite convergence time of random and concurrent best response dynamics, although in each step bilateral best responses are unique and can be found easily.

Theorem 7.3. *There are minimum effort games that have convex functions, uniform budgets, and starting states, from which any dynamics using only bilateral best responses does not converge to a stable state.*

Proof. We consider two paths of length 4 as in the games of Example 4.4 and introduce a new player v_c as shown in Figure 2. All players have budget 2. In our starting state s we assign all incident players to contribute 1 to the edges (z_1, v_c) and (z_2, v_c) . This yields a maximum revenue of 2000 for v_c . As long as this remains the case, v_c will never participate in a bilateral deviation. In turn, in every unilateral best response players z_1 and z_2 will match the contribution of v_c towards their joint edges. Note that this essentially creates the budget restriction for the z -players that is necessary to show non-existence of a pairwise equilibrium in Example 4.4. It remains to show that we can implement the cycling of dynamics in terms of bilateral best responses. For this, note that for player u_1 it is always a unilateral best response to match any contribution of v_1 on their joint edge (similarly for u_2 and v_2). The same is true for z_1 , he will match the contribution of w_1 up to an effort of 1. Finally, the joint deviations of players v_1 and w_1 are bilateral best responses as well. This implies that the cycling dynamics outlined above in Example 4.4 remain present when we restrict to bilateral best responses. Thus, no stable state can be reached. \square

Observe that in this game there are sequences of bilateral deviations that converge to a pairwise equilibrium, but the bilateral deviations are not bilateral best responses. Consider an arbitrary cycling sequence of bilateral deviations from our starting state, and w.l.o.g. consider the cycling dynamics happen on the upper path in Figure 2. Then at some point we will see a bilateral deviation of w_2 and z_2 , in which on their joint edge z_2 contributes 1 and w_2 increases his effort. This creates a strict improvement of utility for both of them. Note that a bilateral deviation allows both w_2 and z_2 to change their strategies in arbitrary manner. Thus, while increasing his contribution towards w_2 , z_2 can also simultaneously decrease his contribution towards v_c . If the decrease is very tiny, the increase in reward on the edge to w_2 outweighs

the decrease of reward on the edge to v_c . In this way, this deviation still generates a strict improvement of utility for z_2 . Hence, both z_2 and w_2 would make strict improvement although z_2 decreases the contribution towards v_c by a tiny amount, hence this represents a profitable bilateral deviation (but obviously not a best response). Afterwards, the balance for v_c is broken, and v_c and z_1 have a bilateral deviation to put all effort on their joint edge. This quickly leads to a pairwise equilibrium. Naturally, the argument works symmetrically for z_1 . However, such an evolution is quite unreasonable, as it is always in the interest of the z -players to keep their contribution towards v_c as high as possible.

Minimum Effort Games and Concave Functions For concave functions, we can use the following simple rule to find a bilateral best response. Consider two incident players u and v in state s connected by edge e . In the first phase we consider each player independently and compute a unilateral best response s'_u and s'_v under the assumption that the other player would match his contribution on e . Then we fix the strategy of the player u for which $s'_u(e) < s'_v(e)$. In the state s' , player u is perfectly happy with his choice and would not participate in any bilateral deviation. However, player v might be willing to deviate, so we recalculate a unilateral best response for v under the condition that $s'_v(e) \leq s'_u(e)$. Note that, due to concavity of the functions, v has a unilateral best response that matches $s'_u(e)$. This yields a pair of mutual best responses: u has best possible utility (even if it were able to control v 's strategy), and v has the best possible utility given u 's strategy. As usual, the players switch to (s'_u, s'_v) if and only if it is a profitable bilateral deviation.

Theorem 7.4. *Random and concurrent best response dynamics converge to a pairwise equilibrium when all reward functions are $f_e(x, y) = h_e(\min(x, y))$ with differentiable and strictly concave h_e .*

Proof. We measure progress in terms of the derivatives of the edges. For a state s consider an edge with highest derivative $e_{max} = \arg \max_{e \in E} h'_e$, where $h'_{(u,v)} = h'_e(\min(s_u(e), s_v(e)))$. Obviously, $h'_{max} = h'_{e_{max}} \geq h'_e$ for any other edge $e \in E$, so in any unilateral or bilateral best response the incident players will not try to remove effort from this edge once $s_u(e) = s_v(e)$. Edge $e_{max} = (u, v)$ is *stabilized* if there is a player u with $h'_{max} = h'_{e'}$ for every edge $e' = (u, v') \in E$ and spending all his budget, i.e., $\sum_{e'=(u,v') \in E} s_{e'}(u) = B_u$. In this case, no player will remove effort from e_{max} , but at least one player has no interest in increasing effort on e_{max} .

We now consider the dynamics starting in a state s and the set of non-stabilized edges E_{max} with maximum derivative h'_{max} among non-stabilized edges. Suppose that a bilateral deviation results in a reduction of the minimum effort on any edge e to a value x with $h'_e(x) > h'_{max}$. This is a contradiction to h'_{max} being the currently highest derivative value and the deviation being composed of mutual unilateral best responses. Hence, the value h'_{max} will never increase over the run of the dynamics. As an edge $e \in E_{max}$ with highest derivative is not stabilized, both incident players have other incident edges with strictly smaller derivative. Hence, if they play a bilateral best response, they strictly increase effort on e while strictly decreasing effort on other edges. By strict concavity this implies that after the step $h'_e < h'_{max}$. In addition, both players picking a best response means that the derivative of all edges that were previously lower than h'_{max} now remain at most h'_e . This means that no new edge with derivative value h'_{max} is created, but e is removed. Thus, in each such step we either increase the number of stabilized edges, or we decrease the number of edges of highest derivative among non-stabilized edges. As such a step is played after a finite number of steps in expectation, this argument proves convergence.

It remains to show that the resulting state, in which all edges are stabilized, is resilient to all unilateral and bilateral deviations and not only against the type of bilateral best responses we used to converge to it. Here we can apply an inductive argument similar to Theorem 4.8 that no profitable bilateral deviation exists and the state is indeed a pairwise equilibrium. Note that the argument simplifies quite drastically for the case of *strictly* concave functions. In particular, we consider the edge with maximum derivative. For at least one incident player v , all edges with positive minimum effort have the same derivative, hence this player will never change his strategy. In addition, the other adjacent players have an incentive to keep their efforts on the edges with v . Thus, we can remove v , reduce the budgets of incident players and iterate. This proves the theorem. \square

The previous proof shows that convergence is achieved in the limit, but the decrease of the maximum derivative value is not bounded. If the state is close to a pairwise equilibrium, the changes could become arbitrarily small, and the convergence time until reaching the exact equilibrium could well be infinite.

8 General Contribution Games

In this section we generalize some of our results to general contribution games. However, a detailed study of such general games remains as an open problem. A general contribution game can be represented by a hypergraph $G = (V, E)$. The set of nodes V is the set of players, and each edge $e \in E$ is a hyperedge $e \subseteq 2^V$ and represents a joint project of a subset of players. Reward functions and player utilities are defined as before. In particular, using the notation $s_e = (s_u)_{u \in e}$ we get reward functions $f_e(s_e)$ with $f_e : (\mathbb{R}_{\geq 0})^{|e|} \rightarrow \mathbb{R}_{\geq 0}$. In this case, we extend our stability concept to *setwise equilibrium* that is resilient against all deviations of all player sets that are a subset of any hyperedge. In a setwise equilibrium no (sub-)set of players incident to the same hyperedge has an improving move, i.e., a combination of strategies for the players such that every player of the subset strictly improves. More formally, a *setwise equilibrium* s is a state such that for every edge $e \in E$ and every player subset $U \subseteq e$ we have that for every possible deviation $s'_U = (s'_u)_{u \in U}$ there is at least one player $v \in U$ that does not strictly improve $w_v(s) \geq w_v(s'_U, s_{-U})$, where $s_{-U} = (s_u)_{u \in V-U}$. Note that this definition includes all unilateral deviations as a special case. The most central parameter in this context will be the size of the largest project $k = \max_{e \in E} |e|$.

We note in passing that Nash equilibria without resilience to multilateral improving moves are again always guaranteed for all reward functions. It is easy to observe that $\Phi(s) = \sum_{e \in E} f_e(s_e)$ is an exact potential function for the game. Note that this is equal to $k \cdot w(s)$ and thus equivalent to $w(s)$ only for uniform hypergraphs, in which all hyperedges have the same cardinality k . In these cases the price of stability for Nash equilibria is always 1. Otherwise, it is easy to construct simple examples, in which all Nash equilibria (and therefore all setwise equilibria) must be suboptimal.⁴

Convex Functions For general convex functions we extend the functions of class \mathcal{C} to multiple dimensions. In particular, the functions are coordinate-convex and have non-negative mixed

⁴Consider a maximum effort game with players u, v_1, v_2 , and v_3 and edges $e = \{u, v_1\}$ and $g = \{u, v_1, v_2\}$. Budgets $B_u = 1$ and $B_{v_i} = 0$ for $i = 1, 2, 3$. Rewards are given by a convex function $h(x)$ for edge g and a function $(1 + \varepsilon)h(x)$ for e . Note that in any Nash equilibrium $s_u(e) = 1$. For small $\varepsilon < 1/2$ this gives $w(s) = (2 + 2\varepsilon)h(1)$, whereas we have higher welfare $w(s) = 3h(1)$ when $s_u(g) = 1$.

partial second derivatives for any pair of dimensions. We first observe that the proof of Theorem 2.5 can be adjusted easily to general games if functions f_e are coordinate-convex and $f_e(s_e) = 0$ whenever $s_u(e) = 0$ for at least one $u \in e$.

Corollary 8.1. *A setwise equilibrium always exists and can be computed efficiently when all reward functions f_e are coordinate-convex and $f_e(s_e) = 0$ whenever $s_u(e) = 0$ for at least one $u \in e$.*

Note that we can also adjust the proof of Claim 2.10 for optimum solutions in a straightforward way. In particular, to obtain the social welfare for projects e , e_1 and e_2 we simply multiply each occurrence of the functions f , f_1 and f_2 in the formulas by the corresponding cardinalities of their edges. This does not change the reasoning and proves an analogous statement of Claim 2.10 also for general games. The actual proof of Theorem 2.9 then is a simple accounting argument that relies on the cardinality of the projects. The observation that the difference between $\sum_e w_e(s^*)$ and $w(s^*)$ is bounded by k yields the following corollary. As previously, these results directly extend to strong equilibria, as well.

Corollary 8.2. *For the class of general contribution games with reward functions in class \mathcal{C} for all $e \in E$ that have a setwise equilibrium, the price of anarchy for setwise equilibria is at most k .*

Minimum Effort Games For minimum effort games some of our arguments translate directly to the treatment of general games. For existence with convex functions and uniform budgets, we can apply the same “greedy matching” argument and wake up players until every hyperedge is incident to at least one awake player. The argument that this creates a setwise equilibrium is almost identical to the one given in Theorem 4.2 for pairwise equilibria. This yields the following corollary.

Corollary 8.3. *A setwise equilibrium always exists in games with uniform budgets and $f_e(s_e) = h_e(\min_{u \in e} s_u(e))$ when all h_e are convex. If all h_e are strictly convex, all setwise equilibria are integral.*

The duality analysis for the price of anarchy in Theorem 4.3 can be carried out as well. In this case, however, the crucial inequality reads $s_e^* \cdot \sum_{u \in e} y_u \geq s_e^* f_e(1) \geq f_e(s_e^*)$. This results in a price of anarchy of k .

Corollary 8.4. *The price of anarchy for setwise equilibria in general contribution games is at most k when all reward functions $f_e(s_e) = h_e(\min_{u \in e} s_u(e))$ with convex h_e , and budgets are uniform.*

Again, both corollaries extend also to strong equilibria.

Maximum Effort Games For maximum effort games it is not possible to extend the main insight in Lemma 5.1 to general games. There are general maximum effort games without setwise equilibria. This holds even for pairwise equilibria in network contribution games, in which the graph G is not simple, i.e., if we allow multiple edges between agents.

Example 8.5. We consider a simple game that in essence implements a Prisoner’s Dilemma. There is a path of four players u , v , w and z , with edges $e_1 = (u, v)$, $e_2 = (v, w)$ and $e_3 = (w, z)$.

In addition, there is a second edge $e_4 = (v, w)$ between v and w . The budgets are $B_u = B_z = 0$ and $B_v = B_w = 1$. The reward functions are $h_{e_1}(x) = h_{e_3}(x) = 3x$, $h_{e_2}(x) = h_{e_4}(x) = 2x$. Note that for v and w it is a unilateral dominant strategy to put all effort on edges e_1 and e_3 , respectively. However, in that case v and w can jointly increase their reward by allocating all effort to e_2 and e_4 , respectively.

For general maximum effort games characterizing the existence and computational complexity of pairwise, setwise, and strong equilibria is an interesting open problem.

9 Conclusions and Open Problems

In this paper we have proposed and studied a simple model of contribution games, in which agents can invest a fixed budget into different relationships. Our results show that collaboration between pairs of players can lead to instabilities and non-existence of pairwise equilibria. For certain classes of functions, the existence of pairwise equilibria is even NP-hard to decide. This implies that it is impossible to decide efficiently if a set of players in a game can reach a pairwise equilibrium. For many interesting classes of games, however, we are able to show existence and bound the price of anarchy to 2. This includes, for instance, a class of games with general convex functions, or minimum effort games with concave functions. Here we are also able to show that best response dynamics converge to pairwise equilibria.

There is a large variety of open problems that stem from our work. The obvious open problem is to adjust our results for the network case to general set systems and general contribution games. While some of our proofs can be extended in a straightforward way, many open problems, most prominently for concave functions, remain.

Another obvious direction is to identify other relevant classes of games within our model and prove existence and tight bounds on the price of anarchy. Another interesting aspect is, for instance, the effect of capacity constraints, i.e., restrictions on the effort that a player can invest into a particular project.

More generally, instead of a total budget a player might have a function that characterizes how much he has to “pay” for the total effort that he invests in all projects. Such “price” functions are often assumed to be linear or convex (e.g., in [14, 39]).

Finally, an intriguing adjustment that we outlined in the introduction is to view the projects as instances of the combinatorial agency framework and to examine equilibria in this more extended model.

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