Chapter 01: Matchings

(Combinatorial Graph Theory, SS 2019)

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SS 2019, RWTH

Announcements

Lecture times:

- Monday, 12:30-14:00, room AH3
- Friday, 12:30-14:00, room AH3

Lecture: Gerhard Woeginger (E1, room 4024)
Instructions: Tim Hartmann (E1, room 4020)

Web-page:
`https://algo.rwth-aachen.de/Lehre/SS19/KG/KG.py`

Matchings

Definitions

Definition

A matching in an undirected graph $G = (V, E)$ is a subset $M \subseteq E$, so that no two edges in M have a vertex in common.

- A matching M is perfect, if every vertex is incident to some edge in M
- For $A \subseteq V$, define $\Gamma(A) := \{v \in V \mid v \text{ is adjacent to some } a \in A\}$

Hall's theorem (Marriage theorem)

Theorem (Philip Hall, 1935)

A bipartite graph $(X \cup Y, E)$ with $|X| = |Y|$ has a perfect matching, if and only if it satisfies the Hall condition

$$\forall A \subseteq X: \quad |\Gamma(A)| \geq |A|$$

- Suppose that $|\Gamma(S)| = |S|$ for some $S \subset X$ with $\emptyset \neq S \neq X$.
  Induct on subgraph induced by $S \cup \Gamma(S)$ and on subgraph induced by $(X - S) \cup (Y - \Gamma(S))$
- Suppose that $|\Gamma(S)| \geq |S| + 1$ for all $S \subset X$ with $\emptyset \neq S \neq X$.
  Induct on some edge $e = [x, y]$ and on subgraph induced by $(X - \{x\}) \cup (Y - \{y\})$

Birkhoff's theorem

Definition

An $n \times n$ matrix is doubly stochastic,
if all entries are non-negative, and if all row-sums and column-sums are 1.

An $n \times n$ matrix is a permutation matrix,
if all entries are 0 or 1, and if every row and column contain exactly one 1.

Lemma

For every $n \times n$ doubly stochastic matrix $A = (a_{ij})$
there exists a permutation $\pi \in S_n$ so that $a_{i,\pi(i)} \neq 0$ for all i.

Theorem (Garrett Birkhoff, 1946)

Every $n \times n$ doubly stochastic matrix $A = (a_{ij})$
is the convex combination of permutation matrices.

Birkhoff: An application

Problem

INSTANCE: n workers and n tasks;
profits $p_{i,j}$ for $1 \leq i, j \leq n$

GOAL: Assign every task i to some worker $\pi(i)$ (one task per worker), so that the total resulting profit is maximized

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} p_{i,j} \cdot x_{i,j} \\ \text{such that} \quad & \sum_{i=1}^{n} x_{i,j} = 1 && \text{for } j = 1, \ldots, n \\ & \sum_{j=1}^{n} x_{i,j} = 1 && \text{for } i = 1, \ldots, n \\ & 0 \leq x_{i,j} \leq 1 && \text{for } i, j = 1, \ldots, n \end{aligned}$$

Note: Variables $x_{i,j}$ are real variables

One-sided vs two-sided Hall

Theorem (Philip Hall, 1935)

A bipartite graph $(X \cup Y, E)$ with $|X| = |Y|$ has a perfect matching, if and only if it satisfies the one-sided Hall condition

$$\forall A \subseteq X: \quad |\Gamma(A)| \geq |A|$$

Theorem

A bipartite graph $(X \cup Y, E)$ has a perfect matching, if and only if it satisfies the two-sided Hall condition

$$\forall A \subseteq X: \quad |\Gamma(A)| \geq |A|$$

$$\forall B \subseteq Y: \quad |\Gamma(B)| \geq |B|$$

Correction: Exercise #4

First paragraph of Exercise #4

Let us try to extend Hall's theorem to bipartite graphs with infinitely many vertices: The two vertex sets in the bipartition are $X = \{x_i | i \in \mathbb{N}\}$ and $Y = \{y_i | i \in \mathbb{N}\}$. The extended Hall condition says that every finite subset $A \subseteq X$ should have at least $|A|$ neighbors in Y, and that every infinite subset $A \subseteq X$ should have infinitely many neighbors in Y.

Replace this extended Hall condition by:

The extended two-sided Hall condition says that

- every finite subset $A \subseteq X$ should have at least $|A|$ neighbors in Y, and every infinite subset $A \subseteq X$ should have infinitely many neighbors in Y;
- every finite subset $B \subseteq Y$ should have at least $|B|$ neighbors in X, and every infinite subset $B \subseteq Y$ should have infinitely many neighbors in X.

Kőnig's theorem

Definition

Let $G = (V, E)$ be a graph.
The matching number $\nu(G)$ is the size of the largest matching.
The vertex cover number $\tau(G)$ is the size of the smallest vertex cover.

A vertex cover is a subset of the vertices, that contains at least one end-point of every edge.

Theorem (Dénes Kőnig, 1931)

Every bipartite graph G satisfies $\nu(G) = \tau(G)$.

Some exercises

Exercise

The front and the back side of an A4 piece of paper are each divided into 10 polygonal regions of equal area. Show that one can pierce the paper with 10 pins so that each region (on either side) is pierced exactly once.

Exercise

(a) Find a 3-regular graph that does not have a perfect matching.
(b) Prove that every bipartite 3-regular graph has a perfect matching.

Exercise

Prove the defect version of Hall's theorem:
Let $G = (X \cup Y, E)$ be a bipartite graph, and let $d \geq 0$ be an integer. If every subset $A \subseteq X$ satisfies $|\Gamma(A)| \geq |A| - d$, then G contains a matching with at least $|X| - d$ edges.

A miraculous algorithm

A miraculous algorithm

```
Repeat 100n^3 log n times
    NormalizeColumns(A);
    If 1 − 1/n < r_i < 1 + 1/n for all i: return YES;
    NormalizeRows(A);
    If 1 − 1/n < c_j < 1 + 1/n for all j: return YES;
EndRepeat
return NO
```

```
NormalizeRows(A)
    For i = 1 to n do r_i := Σ_j a_ij
    For i, j = 1 to n do a_ij := a_ij / r_i
```

```
NormalizeColumns(A)
    For j = 1 to n do c_j := Σ_i a_ij
    For i, j = 1 to n do a_ij := a_ij / c_j
```

Example

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} \rightarrow \begin{matrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 \end{matrix} \rightarrow \begin{matrix} 2/5 & 2/5 & 2/7 \\ 3/5 & 0 & 3/7 \\ 0 & 3/5 & 3/7 \end{matrix}$$

$$\rightarrow \begin{matrix} 7/19 & 7/19 & 5/19 \\ 7/12 & 0 & 5/12 \\ 0 & 7/12 & 5/12 \end{matrix}$$

$$\rightarrow \begin{matrix} 12/31 & 12/31 & 12/50 \\ 19/31 & 0 & 19/50 \\ 0 & 12/31 & 19/50 \end{matrix}$$

Analysis (1)

Theorem

If the miraculous algorithm returns YES,
then the underlying graph has a perfect matching.

Proof: Use scaled adjacency matrix to show that Hall condition is satisfied

Analysis (2)

Theorem

If the underlying graph has a perfect matching,
then the miraculous algorithm returns YES.

Proof: Analyze the permanent $\operatorname{per}(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} a_{i,\pi(i)}$

- After first normalization: $\operatorname{per}(A) \geq 1/n^n$
- All through the algorithm: $\operatorname{per}(A) \leq 1$
- Every normalization multiplies $\operatorname{per}(A)$ by a factor of at least $\exp(1/(6n^2))$

Analysis (3)

Some useful facts

- For $x \in \mathbb{R}$, we have $\ln(1+x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}$
- The product of two (positive) reals x and y with sum $x + y = s$ is maximized at $x = y = s/2$
- Arithmetic-Geometric-Mean inequality (A.G.M):
  For $x_1, \ldots, x_n \in \mathbb{R}_0^+$: $\frac{1}{n}\sum x_i \geq \left(\prod x_i\right)^{1/n}$
- Arithmetic-Quadratic-Mean inequality (A.Q.M):
  For $x_1, \ldots, x_n \in \mathbb{R}_0^+$: $\sqrt{\frac{1}{n}\sum x_i^2} \geq \frac{1}{n}\sum x_i$