Exercise 1:
Let $I$ denote an instance of the knapsack problem with $n$ items with profits $p_1, \ldots, p_n \in \mathbb{N}$ and weights $w_1, \ldots, w_n \in \mathbb{N}$, and let $t \in \mathbb{N}$ denote the capacity of the knapsack. We split the items into two halves and use the Nemhauser/Ullmann algorithm to generate lists $L_1$ and $L_2$ of the Pareto optimal solutions of the knapsack instance that consists only of the items $1, \ldots, \frac{n}{2}$ and $\frac{n}{2} + 1, \ldots, n$, respectively. We assume that these lists are sorted in non-decreasing order of the weights of the solutions.

a) Show that every Pareto optimal solution of the instance $I$ can be obtained by combining two solutions from $L_1$ and $L_2$.

b) In the worst case, the number of Pareto optimal solutions of $I$ is $|L_1| \cdot |L_2|$. Find such a worst-case instance.

c) Show that the optimal solution of $I$ can be found in time $O(|L_1| + |L_2|)$.

Exercise 2:
Let $X_1, \ldots, X_n$ denote independent random variables that are drawn uniformly at random from the interval $[0,1]$. We say that the random variable $X_i$ is a left-to-right maximum if $X_i > X_j$ for all $j < i$. What is the asymptotic expected number of left-to-right maxima?

Exercise 3:
Let $G = (V, E)$ denote an arbitrary undirected graph with $m$ edges. For every edge $e \in E$, we choose a weight $w(e)$ uniformly at random from the set $\{1, \ldots, k\}$. How large do we have to choose $k$ such that the maximum matching is unique with probability at least $1/2$.

Exercise 4:
The 2-Opt heuristic is a local search algorithm for the TSP. It starts with an arbitrary tour and incrementally improves this tour by making successive improvements that exchange two of the edges in the tour with two other edges. More precisely, in each step the 2-Opt algorithm selects two edges $\{u_1, u_2\}$ and $\{v_1, v_2\}$ from the tour such that $u_1, u_2, v_1, v_2$ are distinct and appear in this order in the tour, and the algorithm replaces these edges by the edges $\{u_1, v_1\}$ and $\{u_2, v_2\}$, provided that this change decreases the length of the tour. The algorithm terminates in a local optimum in which no further improving step is possible.

Let $G = (V, E)$ denote a complete graph and assume that the length $l(e)$ of every edge $e \in E$ is chosen independently and uniformly at random from the interval $[0,1]$. Give an upper bound on the expected number of local improvements until a locally optimal solution is reached.